Well-Covered Graphs and Greedoids

Vadim E. Levit[1,2] **Eugen Mandrescu**[2]

[1] Department of Computer Science and Mathematics
Ariel University Center of Samaria,
Ariel 40700, ISRAEL,
Email: levitv@ariel.ac.il

[2] Department of Computer Science
Holon Institute of Technology,
Holon 58102, ISRAEL,
Email: eugen_m@hit.ac.il

Abstract

G is a *well-covered graph* provided all its maximal stable sets are of the same size (Plummer, 1970). S is a *local maximum stable set* of G, and we denote by $S \in \Psi(G)$, if S is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of S.

In 2002 we have proved that $\Psi(G)$ is a greedoid for every forest G. The bipartite graphs and the triangle-free graphs, whose families of local maximum stable sets form greedoids were characterized by Levit and Mandrescu (2003, 2007a).

In this paper we demonstrate that if a graph G has a perfect matching consisting of only pendant edges, then $\Psi(G)$ forms a greedoid on its vertex set. In particular, we infer that $\Psi(G)$ forms a greedoid for every well-covered graph G of girth at least 6, non-isomorphic to C_7.

Keywords: local maximum stable set, greedoid, very well-covered graph, unique perfect matching.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The vertices $x, y \in V(G)$ are called *adjacent* if they are the endpoints of some edge in G, and we write $xy \in E(G)$. We assume also that $xx \notin E(G)$, for every $x \in V(G)$, i.e., G is loopless. If $X \subset V$, then $G[X]$ is the subgraph of G induced by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, i.e., $G - F = (V, E - F)$, and we write shortly $G - e$, whenever $F = \{e\}$.

The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N[v] = \{v\} \cup N(v)$. If $|N(v)| = |\{u\}| = 1$, then v is a *pendant vertex* and vu a *pendant edge* of G. By pend(G) we mean the set of all pendant vertices of G.

K_n, C_n denote, respectively, the *complete graph* on $n \geq 1$ vertices, and the *chordless cycle* on $n \geq 3$ vertices, i.e., $K_1 = (\{v_1\}, \emptyset)$ and

$$K_n = (\{v_i : 1 \leq i \leq n\}, \{v_i v_j : 1 \leq i < j \leq n\}), n \geq 2,$$

while C_n has

$$V(C_n) = \{v_i : 1 \leq i \leq n\},$$

$$E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\}).$$

A vertex $v \in V(G)$ is called *simplicial* if $G[N[v]]$ is a complete subgraph of G.

We denote the *neighborhood* of some $A \subset V$ by $N_G(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and its *closed neighborhood* by $N_G[A] = A \cup N(A)$, or shortly, $N(A)$ and $N[A]$, respectively, if no ambiguity.

A *tree* is a cycle-free connected graph, while a *forest* is cycle-free graph.

A *stable* set in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a *maximum stable set* of G, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G. In the sequel, by $\Omega(G)$ we denote the set of all maximum stable sets of the graph G.

A set $A \subseteq V(G)$ is a *local maximum stable set* of G if A is a maximum stable set in the subgraph induced by $N[A]$, i.e., $A \in \Omega(G[N[A]])$, (Levit and Mandrescu 2002). Let $\Psi(G)$ stand for the set of all local maximum stable sets of G. Notice that $\Omega(G) \subseteq \Psi(G)$ is true for every graph G.

Clearly, every set $S \subseteq$ pend(G) belongs to $\Psi(G)$. Nevertheless, there exist local maximum stable sets that do not contain pendant vertices. For instance, $\{e, g\} \in \Psi(G)$, where G is the graph from Figure 1.

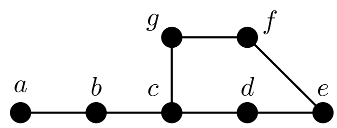

Figure 1: A graph having various local maximum stable sets.

A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of M share a common vertex. A *maximum matching* is a matching of maximum size $\mu(G)$. A matching is *perfect* if it saturates all the vertices of the graph. Let us recall that G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$. It is known that every bipartite graph is a König-Egerváry graph as well.

A graph G is *well-covered* if every maximal stable set of G is also a maximum stable set, i.e., it belongs to $\Omega(G)$. If, in addition, G has no isolated vertices and $|V(G)| = 2\alpha(G)$, then G is *very well-covered* (Favaron 1982). For instance, the graph depicted in Figure 1 is well-covered, but not very well-covered, while the graph from Figure 2 is very well-covered.

In other words, each stable set of a well-covered graph is contained in a maximum stable set, e.g., the

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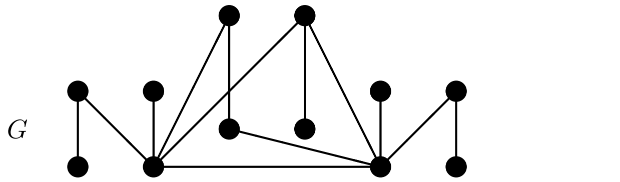

Figure 2: A very well-covered graph whose unique perfect matching has non-pendant edges.

graph H from Figure 3. Since there is no maximum stable set S of G such that $\{b, d\} \subset S$, the graph G in Figure 3 is not well-covered.

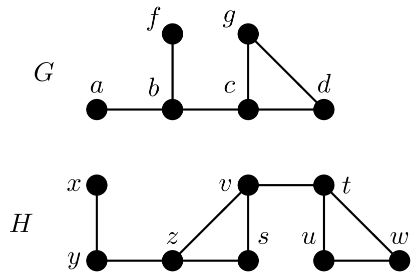

Figure 3: H is well-covered; G is not well-covered.

Well-covered graphs were defined by Plummer (1970). A number of classes of well-covered graphs were completely described; e.g., well-covered bipartite graphs (Ravindra 1977), very well-covered graphs (Favaron 1982), well-covered block graphs and unicyclic graphs (Topp and Volkmann 1990), well-covered graphs of girth ≥ 6 (Finbow, Hartnell and Nowakowski 1993), well-covered cubic graphs (Campbell, Ellingham and Royle 1993), well-covered graphs that contain neither 4- nor 5-cycles (Finbow, Hartnell and Nowakowski 1994), 4-connected claw-free well-covered graphs (Hartnell and Plummer 1996), well-covered simplicial, chordal, and circular arc graphs (Prisner, Topp and Vestergaard 1996), well-covered König-Egerváry graphs (Levit and Mandrescu 1998). A survey on this subject is due to Plummer (1993).

In fact, well-covered graphs are exactly those graphs for which the greedy algorithm constructing maximum stable sets vertex by vertex always yields a maximum stable set, no matter how its greediness makes it to chose vertices of a graph. For general graphs, the problem of finding a maximum stable set, is **NP**-hard.

While, in general, it is co-**NP**-complete to determine if a given graph is well-covered (Chvátal and Slater 1993, Sankaranarayana and Stewart 1992), recognizing weighted well-covered graphs with bounded $\triangle(G)$ can be done in polynomial time (Caro et al. 1998, Zverovich 2004), where $\triangle(G)$ equals the maximum vertex degree of the graph G. Tankus and Tarsi (1996, 1997) showed that claw-free well-covered graphs can be recognized in polynomial time.

It is easy to prove the following.

Proposition 1.1 *Every graph having a perfect matching consisting of pendant edges is very well-covered.*

The converse of Proposition 1.1 is not generally true (e.g., the graph G depicted in Figure 2). Moreover, there are well-covered graphs without perfect matchings, (e.g., K_3). Nevertheless, following Favaron's characterization for very well-covered graphs (i.e., Theorem 1.2), one can assert that "having a perfect matching" is a necessary condition for very well-coveredness.

A matching M in a graph G satisfies *Property P* if for every edge $xy \in M$,

$$N(x) \cap N(y) = \emptyset \text{ and}$$

$$N(x) - \{y\} \text{ is adjacent to all of } N(y) - \{x\}.$$

Theorem 1.2 *For a graph G without isolated vertices the following are equivalent:*
(i) *G is very well-covered;*
(ii) *there is a perfect matching in G that satisfies Property P;*
(iii) *there exists at least one perfect matching in G and every perfect matching in G satisfies Property P.*

By $H \circ K_1$ we mean the graph obtained from H by appending a single pendant edge to each vertex of H. Let us notice that $H \circ K_1$ is very well-covered and $\alpha(H \circ K_1) = |V(H)|$. Moreover, Finbow, Hartnell and Nowakowski (1993) showed (Theorem 1.3) that, under certain conditions, every well-covered graph must be of this form.

Theorem 1.3 *Let G be a connected graph of girth greater than five, which is isomorphic to neither C_7 nor K_1. Then G is well-covered if and only if its pendant edges form a perfect matching, i.e., $G = H \circ K_1$ for some graph H.*

In other words, Theorem 1.3 shows that, apart from K_1 and C_7, connected well-covered graphs of girth ≥ 6 are very well-covered. Consequently, a tree $T \neq K_1$ could be only very well-covered, and this is the case if and only if $T = H \circ K_1$ for some tree H (for additional details, see Ravindra 1977, Favaron 1982, Levit and Mandrescu 1999).

The following theorem concerning maximum stable sets in general graphs, due to Nemhauser and Trotter Jr. (1975), shows that some stable sets can be enlarged to maximum stable sets.

Theorem 1.4 *Every local maximum stable set of a graph is a subset of a maximum stable set.*

The graph W from Figure 1 has the property that every $S \in \Omega(W)$ contains some local maximum stable set, but these local maximum stable sets are of different cardinalities: $\{a, d, f\} \in \Omega(W)$ and $\{a\}, \{d, f\} \in \Psi(W)$, while for $\{b, e, g\} \in \Omega(W)$ only $\{e, g\} \in \Psi(W)$.

However, there exists a graph G satisfying the equality $\Psi(G) = \Omega(G)$, e.g., $G = C_n$, for $n \geq 4$.

A greedoid (Björner and Ziegler 1992, and Korte et al. 1991) is a set system generalizing the notion of matroid.

Definition 1.5 *A greedoid is a pair $(V, \mathcal{F})$, where $\mathcal{F} \subseteq 2^V$ is a non-empty set system satisfying the following conditions:*
(Accessibility) for every non-empty $X \in \mathcal{F}$ there is an $x \in X$ such that $X - \{x\} \in \mathcal{F}$;
(Exchange) for any $X, Y \in \mathcal{F}, |X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

Recall that a matroid is a set system $(V, \mathcal{F})$ that satisfies both the "exchange property" and the "hereditary property", saying that : if $X \in \mathcal{F}$ and $Y \subseteq X$, then $Y \in \mathcal{F}$. Evidently, any matroid is also a greedoid. It is clear that the family of all stable sets of a graph is a matroid if and only if G is a disjoint union of complete graphs, which means that, necessarily, G must be well-covered of a specific form.

If G is well-covered, $\Psi(G)$ is a matroid if and only if each $S \in \Omega(G)$ consists of only simplicial vertices, because $\Omega(G) \subseteq \Psi(G)$ and every $v \in S$, by hereditary property, satisfies $\{v\} \in \Psi(G)$, i.e., $G[N[v]]$ must be a complete graph.

The notion of matroid was defined by Whitney (1935). Later Edmonds (1971)characterized a matroid as a hereditary set system for which a class of linear optimization problems can be solved by greedy algorithms. Korte and Lovász (1991) introduced the greedoid in an attempt to generalize this characterization to accessibility systems.

It is worth mentioning that if $(V, \mathcal{F})$ is a greedoid and $X \in \mathcal{F}$, $|X| = k \geq 2$, then according to accessibility property, one can build a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset \{x_1, ..., x_3\} \subset ...$$

$$... \subset \{x_1, ..., x_{k-1}\} \subset \{x_1, ..., x_{k-1}, x_k\} = X$$

such that $\{x_1, ..., x_j\} \in \mathcal{F}$, for each $j \in \{1, ..., k-1\}$. Such a chain we call an *accessibility chain* of X.

For example, $\Psi(G_1)$ is a greedoid and

$$\{a\} \subset \{a, b\} \subset \{a, b, c\}$$

is an accessibility chain of $\{a, b, c\} \in \Psi(G_1)$, where G_1 is presented in Figure 4.

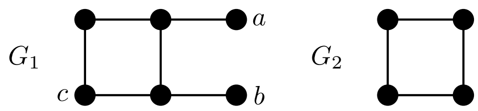

Figure 4: G_1, G_2 are very well-covered graphs, but only for G_1 it is true that every $S \in \Omega(G_1)$ has an accessibility chain.

Levit and Mandrescu (2002) proved the following.

Theorem 1.6 *For every forest T, $\Psi(T)$ is a greedoid on its vertex set.*

The case of bipartite graphs having a unique cycle, whose family of local maximum stable sets forms a greedoid, is studied in Levit and Mandrescu (2001, 2005). The general case of bipartite graphs was treated in Levit and Mandrescu (2003), while for triangle-free graphs we refer the reader to Levit and Mandrescu (2007) for details. Nevertheless, there exist non-bipartite and non-triangle-free graphs whose families of local maximum stable sets form greedoids. The families $\Psi(G_1), \Psi(G_2), \Psi(G_3), \Psi(G_4)$ of the graphs in Figure 5 are greedoids. Let us notice that G_1 is very well-covered and G_3 is well-covered, while G_2, G_4 are not well-covered and also non-triangle-free.

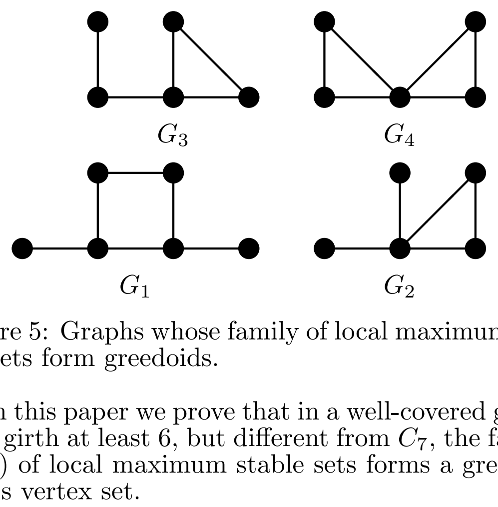

Figure 5: Graphs whose family of local maximum stable sets form greedoids.

In this paper we prove that in a well-covered graph G of girth at least 6, but different from C_7, the family $\Psi(G)$ of local maximum stable sets forms a greedoid on its vertex set.

2 Results

It is easy to see that no maximum stable set of C_7 admits an accessibility chain. The graph G in Figure 6 shows that even if some $S \in \Omega(G)$ admits an accessibility chain, this is not necessarily true for all maximum stable sets. The case of the graph H from Figure 6 is different: each maximum stable set of H has an accessibility chain, and the reason is given in Proposition 2.1.

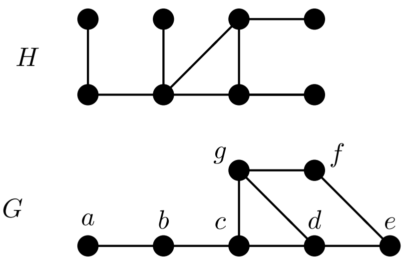

Figure 6: $\{a, c, f\}, \{a, g, e\} \in \Omega(G)$, but only $\{a, c, f\}$ admits an accessibility chain.

Proposition 2.1 *Every maximum stable set of the graph $G = H \circ K_1$ has an accessibility chain.*

Proof. Clearly, $\alpha(G) = n$, where $|V(H)| = n$, and each $S \in \Omega(G)$ satisfies $S \cap \text{pend}(G) \neq \emptyset$.

We prove by induction on n that every $S \in \Omega(G)$ has an accessibility chain.

For $n = 1$, the assertion is clearly true.

For $n = 2$, let $S = \{x_1, x_2\} \in \Omega(G)$. Then at least one of x_1, x_2 is pendant, say x_1. Hence, the chain is $\{x_1\} \subset \{x_1, x_2\} = S$.

Suppose that the assertion is true for $k < n$.

Let $G = (V, E) = H \circ K_1$ be with $|V(H)| = n$, and let $S \in \Omega(G)$.

Since $S \cap \text{pend}(G) \neq \emptyset$, let $a_1 \in S \cap \text{pend}(G)$. If $N(a_1) = \{b_1\}$, then $G - \{a_1, b_1\} = (H - \{b_1\}) \circ K_1$. Hence, we have that

$$S_{n-1} = S - \{a_1\} \in \Omega(G - \{a_1, b_1\}),$$

and by induction hypothesis, there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, x_2, ..., x_{n-1}\} = S_{n-1}$$

such that $\{x_1, x_2, ..., x_k\} \in \Psi(G - \{a_1, b_1\})$ for each $k \in \{1, ..., n-1\}$. Since $N_G(a_1) = \{b_1\}$, it follows that

$$N_G(\{x_1, x_2, ..., x_k\} \cup \{a_1\}) =$$
$$= N_{G-\{a_1,b_1\}}(\{x_1, x_2, ..., x_k\}) \cup \{b_1\},$$

and therefore $\{x_1, x_2, ..., x_k\} \cup \{a_1\} \in \Psi(G)$, for every $k \in \{1, ..., n-1\}$. Clearly, $\{a_1\} \in \Psi(G)$, and consequently, we obtain the chain:

$$\{a_1\} \subset \{a_1, x_1\} \subset \{a_1, x_1, x_2\} \subset ...$$

$$... \subset \{a_1, x_1, x_2, ..., x_{n-1}\} = \{a_1\} \cup S_{n-1} = S,$$

such that $\{a_1, x_1, x_2, ..., x_k\} \in \Psi(G)$ for every k from $\{1, ..., n-1\}$, i.e., S has an accessibility chain. ■

Let us notice that Proposition 2.1 is not valid for each very well-covered graph; e.g., C_4 is very well-covered, but no $S \in \Omega(C_4)$ has an accessibility chain.

Remark 2.2 *If S consists of only isolated vertices of H, then $S \in \Psi(H \circ K_1)$, because, in this case, $S \subseteq \text{pend}(G)$.*

Remark 2.3 *If S is stable in H and $N_H(S) \neq \emptyset$, then $S \notin \Psi(H \circ K_1)$, because for each $a \in N_H(S)$, the set $\{a\} \cup \{u : u \in \text{pend}(H \circ K_1) \cap N_{H \circ K_1}(S)\}$ is stable in $H \circ K_1$ and larger than S.*

Remark 2.4 *If v is an isolated vertex of the graph H and $S \in \Psi(H \circ K_1)$, such that $S \cap N_{H\circ K_1}[v] = \emptyset$, then $S \cup \{v\} \in \Psi(H \circ K_1)$.*

Lemma 2.5 *If H has no isolated vertices and S is a stable set in $G = H \circ K_1$, then the following assertions are equivalent:*

(i) *$S \in \Psi(G)$;*

(ii) *$S = S_1 \cup S_2$, where $\emptyset \neq S_1 \subseteq \text{pend}(G)$ and $S_2 \subseteq V(H), N_H(S_2) \subseteq N_G(S_1)$;*

(iii) *$G[N_G[S]] = H' \circ K_1$, for some subgraph H' of H, and $S \in \Omega(H' \circ K_1)$.*

Proof. Let us denote:

$$V(H) = \{v_i : 1 \leq i \leq n\},$$

$$V(G) = V(H) \cup \{u_i : 1 \leq i \leq n\},$$

$$E(G) = E(H) \cup \{u_i v_i : 1 \leq i \leq n\}.$$

Notice that

$$\alpha(G) = n, S_0 = \{u_i : 1 \leq i \leq n\} \in \Omega(G)$$

and S_0 =pend(G), since H has no isolated vertices.

(i) $\Longrightarrow$ (ii) Assume that $S \in \Psi(G)$.

Let $S_1 = S \cap \text{pend}(G)$ and $S_2 = S \cap V(H)$. Clearly, $S_1 \neq \emptyset$, because S has an accessibility chain.

If $S_2 = \emptyset$, then the assertion is clearly true.

Suppose that $S_2 \neq \emptyset$. If $N_H(S_2) \nsubseteq N_G(S_1)$, then there must be some $v_k \in N_H(S_2)$ such that $u_k \notin S$, i.e., $u_k \notin S_1$. Hence, we get that

$$\{u_k\} \cup (N_G[S] - V(H))$$

is a stable set in $G[N_G[S]]$ larger than S, in contradiction with $S \in \Psi(G)$.

(ii) $\Longrightarrow$ (iii) Let $S_3 = \{u_k : v_k \in S_2\}$. Then we infer that

$$G[N_G[S]] = G[S_1 \cup S_3] = H' \circ K_1,$$

for some subgraph H' of H. In addition, we have also that

$$|S| = |S_1| + |S_2| = |S_1| + |S_3| \text{ and } S_1 \cup S_3 \in \Omega(G[S]).$$

Consequently, we deduce that $S \in \Omega(H' \circ K_1)$ as well.

(iii) $\Longrightarrow$ (i) As $S \in \Omega(G[N_G[S]])$, it follows, by definition, that $S \in \Psi(G)$. ■

Now we are able to prove the main result of the paper.

Theorem 2.6 *The family $\Psi(H \circ K_1)$ is a greedoid.*

Proof. Let $G = H \circ K_1$ and $S_0 \in \Psi(G)$, i.e., S_0 is a maximum stable set, of size say q, in $H_0 = G[N[S_0]]$.

According to Lemma 2.5, $G[N[S_0]] = H_{S_0} \circ K_1$ for some subgraph H_{S_0} of H, and by Proposition 2.1, we infer that there exists a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset x_1, x_2, x_3\} \subset ...$$

$$... \subset \{x_1, x_2, ..., x_{q-1}\} \subset \{x_1, x_2, ..., x_{q-1}, x_q\} = S_0,$$

such that all $S_k = \{x_1, x_2, ..., x_k\}, 1 \leq k \leq q$, are local maximum stable sets in H_0. Since $N_{H_0}[S_k] = N_G[S_k]$, it results that $S_k \in \Psi(G)$, for any $k \in \{1, ..., q\}$. In other words, $\Psi(G)$ satisfies the accessibility property.

We have to show now that $\Psi(G)$ satisfies also the exchange property.

Let us consider $X, Y \in \Psi(G)$ be such that

$$|Y| = |X| + 1 = m + 1.$$

According to Lemma 2.5*(ii)*, the sets X and Y can be decomposed as follows:

$$X = X_1 \cup X_2 \text{ and } Y = Y_1 \cup Y_2,$$

where X_1, X_2, Y_1, Y_2 satisfy the corresponding conditions, i.e., X_1 and Y_1 are non-empty subsets of pend(G), while X_2, Y_2 are subsets of $V(H)$, such that $N_H(X_2) \subseteq N_G(X_1)$ and $N_H(Y_2) \subseteq N_G(Y_1)$.

Since Y is stable, $X \in \Psi(G)$, and $|X| < |Y|$, it follows that there exists some $y \in Y - X$, such that $y \notin N_G[X]$. In particular, it means that $X \cup \{y\}$ is stable. To check whether $X \cup \{y\} \in \Psi(G)$, we have to analyze the two following cases (see Figure 7).

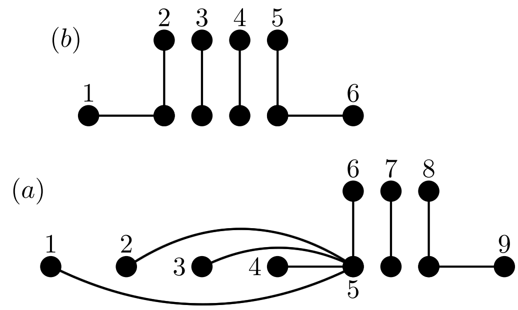

Figure 7: X and Y are local maximum stable sets that illustrate the cases *1* and *2*, respectively. *(a)* $Y = \{1, 2, 3, 4, 6\}, X = \{6, 7, 8, 9\}$, having their upper parts $Y_1 = \{6\} \subset X_1 = \{6, 7, 8\}$; and *(b)* $Y = \{1, 2, 3, 4\}, X = \{4, 5, 6\}$, with their upper parts $Y_1 = \{2, 3, 4\} \nsubseteq X_1 = \{4, 5\}$.

Case 1. $Y_1 \subseteq X_1$.

Firstly, we deduce that $y \in Y_2$. Lemma 2.5*(ii)* implies that $N_H(y) \subseteq N_G(Y_1)$. Since $Y_1 \subseteq X_1$, it follows that $N_G(Y_1) \subseteq N_G(X_1)$. Hence, we get $N_H(y) \subseteq N_G(X_1)$. Therefore, we have that

$$X_1 \subseteq \text{pend}(G), \ X_2 \cup \{y\} \subseteq V(H),$$

and

$$N_H(X_2 \cup \{y\}) = N_H(X_2) \cup N_H(\{y\}) \subseteq N_G(X_1).$$

Consequently, according to Lemma 2.5*(ii)*, we may infer that the stable set $X \cup \{y\}$ is, actually, a maximum local stable set in G.

Case 2. $Y_1 \nsubseteq X_1$.

In this situation, one can choose as y every vertex $z \in Y_1 - X_1$, because clearly, both conditions

$$z \in Y - X \text{ and } X \cup \{z\} \in \Psi(G)$$

are satisfied.

Therefore, $\Psi(G)$ satisfies the exchange property as well.

In conclusion, $\Psi(G)$ is a greedoid on the vertex set of G. ■

Let us notice that $\Psi(C_7)$ is not a greedoid, because every $S \in \Psi(C_7)$ has $|S| \neq 1$.

Corollary 2.7 *Let G be a well-covered graph of girth greater than five, which has no connected components isomorphic to C_7. Then $\Psi(G)$ is a greedoid on the vertex set of G.*

Proof. Firstly, if $G = K_1 = (\{a\}, \emptyset)$, then $\Psi(K_1) = \{\{a\}\}$ and it is clearly a greedoid.

Secondly, if G is a connected well-covered graph of girth ≥ 6, isomorphic to neither C_7 nor K_1, then Theorem 1.3 implies that $G = H \circ K_1$ for some graph H. Further, according to Theorem 2.6, $\Psi(G)$ is a greedoid.

If G is disconnected, and G_i, $i \in \{1, ..., q\}$, are its connected components, then clearly,

$$\Psi(G) = \Psi(G_1) \cup \Psi(G_2) \cup ... \cup \Psi(G_q).$$

and, to complete the proof, one has to take care of every connected component G_i, independently. ■

3 Conclusions

We showed that $\Psi(G)$ is a greedoid on the vertex set of a well-covered graph G, which is well-covered of girth ≥ 6 and non isomorphic to C_7. Since C_5 is well-covered, while $\Psi(C_5)$ is not a greedoid, one can ask to characterize well-covered graphs of girth ≤ 5, whose families of local maximum stable sets form greedoids.

Recently, as proved by Levit and Mandrescu (2007b), each very well-covered graph G of girth ≥ 5 must be of the form $G = H \circ K_1$ for some graph H. Therefore, Corollary 2.7 is true for very well-covered graphs of girth ≥ 5.

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