

Multilayer Grid Embeddings of Iterated Line Digraphs

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Abstract

In this paper, we show that for any fixed d -regular digraph G , every iterated line digraph $L^k(G)$ ($k \geq 1$) can be embedded in d layers using $O(n^2)$ area, where n is the number of vertices in $L^k(G)$. Also, we present $\Omega(\frac{n^2}{\log^2 n})$ lower bound on the area of $L^k(G)$ for any fixed number of layers. Besides, we apply the results to specific families of iterated line digraphs such as de Bruijn digraphs, Kautz digraphs, and wrapped butterfly digraphs.

Keywords: Multilayer grid embedding, Iterated line digraphs, VLSI-layout, Interconnection networks, Bisection width.

1 Introduction

1.1 Multilayer Grid Embeddings

As a VLSI-layout model, Aggarwal *et al.* (1991) defined the k -PCB model (PCB is an abbreviation for a printed circuit board). Consider k stacked grid layers. In the k -PCB model, an embedding of a graph consists of a placement of the vertices as horizontal line segments in grids so that each horizontal line segment is placed in the same position on each grid layer, and drawing of each edge as a path in grids (along grid edges) so that there is no crossing. A path corresponding to an edge is not necessary in a single layer. It can change the layer to the upper or lower layer at contact cuts. An embedding of a graph in the k -PCB model is called a k -layer grid embedding of the graph. The area of a multilayer grid embedding of a graph is the minimum size of a grid which covers the embedding. Although the k -PCB model originally permits the existence of contact cuts, Aggarwal *et al.* (1991) mentioned that the existence of contact cuts is undesirable from a practical point of view. Thus, in this paper, we only treat multilayer grid embeddings *without contact cuts*. That is, any edge is drawn as a grid-path in a single grid layer. Throughout the paper, multilayer grid embeddings mean multilayer grid embeddings without contact cuts.

Figure 1 illustrates an example of a 2-layer grid embedding of a graph, where each line style indicates a layer in which an arc is drawn. Multilayer grid embedding is related to the three-dimensional orthogonal graph drawing (Biedl, Thiele & Wood 2005). In fact, we can define a multilayer grid embedding as

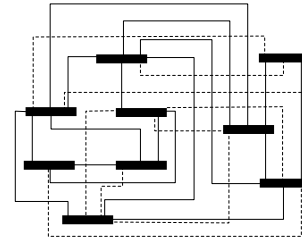


Figure 1: A 2-layer grid embedding of a graph.

a three-dimensional orthogonal graph drawing with several restrictions.

The *thickness* of a graph is the minimum number k such that the graph is a union of k planar graphs. Aggarwal *et al.* (1991) showed the following results. (In what follows, we use n to denote the number of vertices in a graph.)

- Every graph with thickness at most 2 can be embedded in two layers in $O(n^2)$ area.
- There is a graph with thickness at most 2 which needs $\Omega(n^2)$ area for k -layer embedding, where k is any fixed integer.
- Every graph with thickness at most t can be embedded in t layers in $O(n^3)$ area with respect to prespecified placements of the vertices.

Also, Aggarwal *et al.* (1991) proved $\Omega(n^3)$ lower bound on the area of permutation layouts with respect to prespecified placements of the vertices. From this result, the above $O(n^3)$ bound is optimal for the area with respect to prespecified placements of the vertices. In general setting, it still remains open whether any graph with thickness at most t can be embedded in t layers in $O(n^2)$ area.

1.2 Iterated Line Digraphs

Let G be a digraph. A digraph can have loops and symmetric arcs but not multiple arcs. The vertex set and the arc set of G are denoted by $V(G)$ and $A(G)$, respectively. For $v \in V(G)$, the number of arcs with tail (respectively, head) v in G is the outdegree (respectively, indegree) of v . A d -regular digraph is a digraph in which for every vertex v , both the outdegree and the indegree of v are d . The *line digraph* $L(G)$ of G is defined as follows. The vertex set of $L(G)$ is the arc set of G , i.e., $V(L(G)) = A(G)$. The vertex (u, v) is a predecessor of every vertex of the form (v, w) , i.e., $A(L(G)) = \{((u, v), (v, w)) \mid (u, v), (v, w) \in A(G)\}$. When we regard “ L ” as an operation on digraphs, it is called the *line digraph operation*. The k -iterated

line digraph $L^k(G)$ of G is the digraph obtained from G by iteratively applying the line digraph operation k times. Figure 2 shows an example of iterated line digraphs.

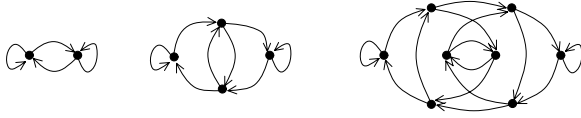


Figure 2: A digraph G , the line digraph $L(G)$, and the 2-iterated line digraph $L^2(G)$.

The line digraph operation is a useful tool for constructing large digraphs with bounded degree, small diameter (Fiol, Yebra & Alegre 1984), and high connectivity (Fábrega & Fiol 1989). In fact, the class of iterated line digraphs of a regular digraph contains well-known interconnection networks for massively parallel computers such as de Bruijn digraphs, Kautz digraphs (Bermond & Peyrat 1989) and wrapped butterfly digraphs (Leighton 1992, Xu 2001).

1.3 Our Results

In this paper, we treat a class of iterated line digraphs of a regular digraph and study on multilayer grid embeddings of such a class. A multilayer grid embedding of a digraph can be similarly defined as that of the underlying graph.

Let G be a d -regular digraph. Then any iterated line digraph $L^k(G)$ is also d -regular. A d -regular digraph can be decomposed into d 1-regular digraphs, and a 1-regular digraph is clearly planar. Thus, the thickness of $L^k(G)$ is at most d . By the result of Aggarwal *et al.* (1991), we can see that $L^k(G)$ can be embedded in d layers in $O(n^3)$ area. In this paper, we show that such an upper bound can be improved, i.e., any iterated line digraph $L^k(G)$ can be embedded in d layers in $O(n^2)$ area. Also, we present $\Omega(\frac{n^2}{\log^2 n})$ lower bound on the area of $L^k(G)$ for any fixed number of layers. In order to derive the lower bound, we show $\Omega(\frac{n}{\log n})$ lower bound on the bisection width of $L^k(G)$, which deserves an independent interest, since the bisection width of a network is an important indicator of its power as a computer network (Bornstein, Litman, Maggs & Sitaraman 2001, Leighton 1992).

This paper is organized as follows. The basic definitions and terminology are given in Section 2. In Section 3, we present an algorithm to embed $L^k(G)$ in $O(n^2)$ area. In Section 4, we show a $\Omega(\frac{n^2}{\log^2 n})$ lower bound on the area for an embedding of $L^k(G)$. In Section 5, we apply the algorithm in Section 3 to specific families of iterated line digraphs. Finally, Section 6 concludes the paper with some remarks.

2 Preliminaries

Let $G = (V, A)$ be a digraph. If $(u, v) \in A(G)$, then u is the *tail* of (u, v) ; v is the *head* of (u, v) . If $u, v \in V(G)$, then u is a *predecessor* of v if $(u, v) \in A(G)$, and u is a *successor* of v if $(v, u) \in A(G)$. The set of arcs with tail u is $A_G^+(u) = \{(u, v) \in A(G)\}$, while the set of arcs with head u is $A_G^-(u) = \{(v, u) \in A(G)\}$.

A *walk* of length k in G is a sequence of vertices (v_0, v_1, \dots, v_k) , where $(v_{i-1}, v_i) \in A(G)$ for each i satisfying $1 \leq i \leq k$. A *path* is a walk in which no

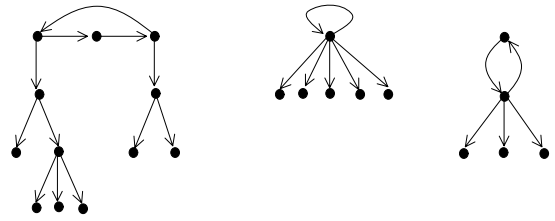


Figure 3: A cycle-rooted tree, a loop-star, and a 2-cycle-star.

vertex is repeated. Let $u, v \in V(G)$. The *distance* $d_G(u, v)$ from u to v in G is the minimum length of a path from u to v in G . The *diameter* of G , denoted by $\text{diam}(G)$, is the maximum distance for any two vertices in G , i.e., $\text{diam}(G) = \max_{u, v \in V(G)} d_G(u, v)$.

The *underlying graph* $U(G)$ of G is a graph obtained from G by replacing each arc with a corresponding edge. Thus, $U(G)$ has loops (respectively, multiple edges with multiplicity two) if G has loops (respectively, symmetric arcs). A digraph is called *weakly connected* if the underlying graph is connected. A (directed) *tree* is a weakly connected digraph in which there is a unique vertex with indegree 0, called the *root*, such that all other vertices have indegree 1. The *depth* of a tree is the maximum length of a path from the root to a vertex with outdegree 0. A (directed) *star* is a tree with depth 1. A *cycle-rooted tree* is a weakly connected digraph in which every vertex has indegree 1. In particular, a digraph obtained from a star and a cycle of length k by identifying the root of the star and a vertex in the cycle is called a *k -cycle-star*. Besides, 1-cycle-star is simply called a *loop-star* (see Figure 3).

Except for marginal grid-points, the degree of every grid-point is four. The upper (respectively, lower) vertical edge incident on a grid-point p is *N -grid-edge* (respectively, *S -grid-edge*) of p . The left (respectively, right) horizontal edge incident on p is *W -grid-edge* (respectively, *E -grid-edge*) of p .

In a multilayer grid embedding of G , each vertex is placed as a horizontal line segment in the same position on each layer. We denote by $\text{lseg}(v)$ the horizontal line segment corresponding to a vertex v of G . The *length* of a horizontal line segment is defined as the number of grid points that are contained in the line segment in a grid. In a multilayer grid embedding of G , each arc is drawn as a path in a grid. We denote by $\text{gpath}(u, v)$ the drawn path corresponding to an arc (u, v) of G . The *starting grid-point* of $\text{gpath}(u, v)$ is the grid-point in $\text{lseg}(u)$ which is incident on a grid-edge used in $\text{gpath}(u, v)$.

3 An Upper Bound on the Area

Theorem 3.1 *For any fixed d -regular digraph G , every $L^k(G)$ ($k \geq 1$) can be embedded in d layers in $O(n^2)$ area, where n is the number of vertices in $L^k(G)$.*

Proof. We present an algorithm for a d -layer grid embedding of $L^k(G)$ for all $k \geq 1$. Let G be a d -regular digraph with p vertices. For each vertex v in G , color the d arcs in $A_G^-(v)$ using d colors. Let F_1, F_2, \dots, F_d be the d subdigraph induced by arcs of the same color. Each F_i has the property that every vertex in F_i has indegree one. This means that each F_i is a disjoint union of cycle rooted trees. Clearly, a cycle-rooted tree is planar. Thus, each F_i is also planar.

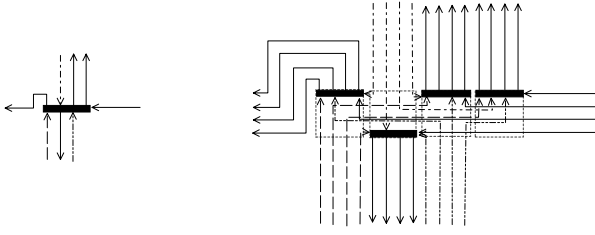


Figure 4: Parts of 4-layer grid embeddings of G and $L(G)$.

Now consider a d -layer grid embedding of G under the following conditions;

- every arc in F_i is drawn as a grid-path in the i -th grid,
- for every vertex v of G , the length of $\text{lseg}(v)$ is d , and for any two arcs (v, u_1) and (v, u_2) in $A_G^+(v)$, $\text{gpath}(v, u_1)$ and $\text{gpath}(v, u_2)$ have distinct starting grid-points in $\text{lseg}(v)$.
- for any arc (v, u) in $A_G^+(v)$, $\text{gpath}(v, u)$ does not contain the W -grid-edge of the leftside grid-point in $\text{lseg}(v)$ and does not contain the E -grid-edge of the rightside grid-point in $\text{lseg}(v)$.

The area of such an embedding of G is not essential for the proof of the theorem, since we consider the class $\{L^k(G) \mid k \geq 1\}$, where G is any fixed regular digraph. Thus, for example, such an embedding can be obtained heuristically. The left figure in Figure 4 is an example of a part of a 4-layer grid embedding of G , where each line-style indicates a layer in which an arc is drawn.

Based on the embedding of G , we construct an embedding of $L(G)$ as follows.

1. Magnify the scale of the embedding of G ; the height d times and the width d times. Each grid-point in a grid is magnified to a rectangular of size $d \times d$. Thus, a horizontal line segment is magnified to a rectangular of size $d \times d^2$.
2. For each (v, u) in $A(G)$ (which is a vertex in $L(G)$), place the corresponding horizontal line segment of length d in the area of the rectangular magnified from the starting grid-point of $\text{gpath}(v, u)$. If $\text{gpath}(v, u)$ use the N -grid-edge (respectively, S -grid-edge) of the starting grid-point in the embedding of G , then the upper-horizontal (respectively, lower-horizontal) line segment in the area of the rectangular is used for $\text{lseg}(v, u)$.

By the above manipulation, we define the placement of the vertices in $L(G)$. Next, we consider the drawing of the arcs in $L(G)$.

3. For drawing of the arcs in $L(G)$, replace each magnified grid-path with a set of d parallel grid-paths. For the magnified grid-path of $\text{gpath}(v, u)$, we will make the set of d parallel grid-paths correspond to the set of d arcs in $\{(v, u), (u, w) \mid (u, w) \in A(G)\}$. Since every grid-path in the set is correctly incident on $\text{lseg}(v, u)$, it remains to join to their head line segments, i.e., $\text{lseg}(u, w)$ for $(u, w) \in A(G)$. Each line segment $\text{lseg}(u, w)$ is placed in the area of the rectangular magnified from $\text{lseg}(u)$.

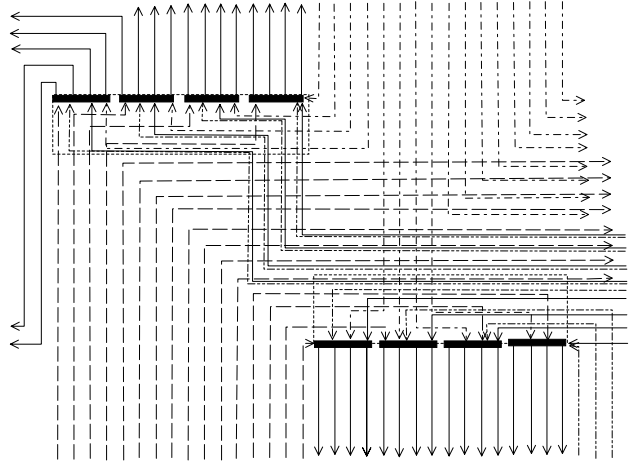


Figure 5: A part of the 4-layer grid embedding of $L^2(G)$ (corresponding to a left-half of the part of embedding of $L(G)$ in Fig.4).

The rectangular magnified from $\text{lseg}(u)$ has the size $d \times d^2$. Thus, it can be checked that we can join d parallel arcs to the d line segments $\text{lseg}(u, w)$, $(u, w) \in A(G)$ in one-to-one manner without crossing (see Figure 4).

By the above manipulation, no crossing in each grid is produced, since the drawing of arcs is based on the drawing of arcs in G . Thus, we obtain a d -layer grid embedding of $L(G)$. By iteratively applying the similar magnifying technique, a d -multilayer grid embedding of $L^k(G)$ for all $k \geq 1$ is obtained (see Figure 5).

Let A be the area of a d -layer grid embedding of G . In the i -th iterative step, we obtain $d^{2i}A$ -area embedding of $L^i(G)$ using d layers. Since $n = |V(L^i(G))| = pd^i$, where $p = |V(G)|$, it holds that $d^{2i}A = \frac{A}{p}n^2 = O(n^2)$. (Note that A and p can be treated as constants, since we consider the class $\{L^k(G) \mid k \geq 1\}$ for any fixed digraph G .) \square

4 A Lower Bound on the Area

In order to derive a lower bound on the area, we use the following result by Aggarwal *et al.* (1991).

Theorem 4.1 (Aggarwal, Klawe & Shor 1991)
Let $c(G)$ be the crossing number of G . Then, for any fixed t , every t -layer grid embedding of G requires $\Omega(c(G))$ area.

A cut (V_1, V_2) of a graph $H = (V, E)$ is the set of edges between V_1 and V_2 , i.e., $\{uv \in E(H) \mid u \in V_1, v \in V_2\}$, where $V_1 \cup V_2 = V(H)$ and $V_1 \cap V_2 = \emptyset$. The bisection width of H is $\min_{|V_1| \geq n/3, |V_2| \geq n/3} |(V_1, V_2)|$. (There is another definition slightly different from this definition of the bisection width (Bornstein *et al.* 2001). However, our result correctly holds for such a definition.)

Pach *et al.* (1996) presented the following relation between the crossing number and the bisection width.

Theorem 4.2 (Pach, Shahrokhi & Szegedy 1996)
Let $c(G)$ and $b(G)$ be the crossing number and the bisection width of G , respectively. Also, let d_1, d_2, \dots, d_n be the degree sequences of G . Then, $b(G)^2 \leq (1.58)^2 (16c(G) + \sum_{i=1}^n d_i^2)$.

The bisection width of a digraph is similarly defined as that of the underlying graph of the digraph. In what follows, we derive a lower bound on the bisection width of $L^k(G)$. (Precisely, simple graphs are assumed in Theorem 4.2. On the other hand, the underlying graph $U(L^k(G))$ may not be simple. However, multiplicity of a multiple edge in $U(L^k(G))$ is at most two. Also, the numbers of loops and multiple edges in $U(L^k(G))$ depend only on G . Thus, our discussion on digraphs below can be combined with the above theorem.)

Theorem 4.3 *Let G be a d -regular digraph. The bisection width of $L^k(G)$ is $\Omega(\frac{n}{\log n})$.*

Proof. Let G_1 and G_2 be digraphs with the same number of vertices. An embedding of G_1 to G_2 consists of a one-to-one mapping f from $V(G_1)$ to $V(G_2)$ and a mapping of each arc (u, v) in G_1 to a path from $f(u)$ to $f(v)$ in G_2 . The complete symmetric digraph K_n^* is an n -vertex digraph in which for any ordered pair (u, v) of vertices, there is an arc from u to v .

Let G be a d -regular digraph with p vertices. In what follows, we will show that there is an embedding of K_n^* , where $n = pd^k = |V(L^k(G))|$, into $L^k(G)$ so that for any arc e of $L^k(G)$, there are at most $O(n \log n)$ paths corresponding to arcs of K_n^* that contain e . This means that the bisection width of $L^k(G)$ is $\Omega(n^2/n \log n) = \Omega(n/\log n)$, since the bisection width of K_n^* is $\Omega(n^2)$.

As a one-to-one mapping from $V(K_n^*)$ to $V(L^k(G))$, we employ any one-to-one mapping. Let ρ be the employed mapping. For an arc $(v_i, v_j) \in A(K_n^*)$, we map it to a shortest path from $\rho(v_i)$ to $\rho(v_j)$ in $L^k(G)$. This mapping from $A(K_n^*)$ to the set of paths in $L^k(G)$ is denoted by ρ^* . By applying the line digraph operation to a digraph one time, the diameter of the digraph increase by at most one (Fiol et al. 1984). Thus, $\text{diam}(L^k(G))$ is at most $\text{diam}(G) + k$. Thus, the number of paths of length at most $\text{diam}(G) + k$ which contain e is an upper bound on the cardinality of the set $\{(u_i, u_j) \in A(K_n^*) \mid e \in \rho^*((u_i, u_j))\}$.

Let $e = (u, v)$. Consider a path of length i which contains e as the k -th arc. Such a path is uniquely decomposed into a path of length $k - 1$ which ends at u , the arc e , and a path of length $(i - k)$ which starts at v . Since G is d -regular, the numbers of paths of length $k - 1$ which ends at u and the paths of length $(i - k)$ which starts at v are at most d^{k-1} and d^{i-k} , respectively. Thus, the number of paths of length i which contains e as the k -th arc is at most $d^{k-1}d^{i-k} = d^{i-1}$. Therefore, the number of paths of length i which contains e is at most d^{i-1} . Hence, the number of paths of length at most $\text{diam}(G) + k$ which contains e is at most $\sum_{i=1}^{\text{diam}(G)+k} d^{i-1}$. Here,

$$\begin{aligned} \sum_{i=1}^t d^{i-1} &= 1 + 2d + 3d^2 + \dots + td^{t-1}, \\ &= (1 + d + d^2 + \dots + d^{t-1}) \\ &\quad + (d + d^2 + \dots + d^{t-1}) + \dots + d^{t-1} \\ &= \frac{d^t - 1}{d - 1} + d \frac{d^{t-1} - 1}{d - 1} + \dots + d^{t-1} \\ &= \frac{1}{d - 1} \sum_{i=0}^{t-1} d^i (d^{t-i} - 1) \\ &= \frac{1}{d - 1} \left(td^t - \frac{d^t - 1}{d - 1} \right). \end{aligned}$$

Thus, $\sum_{i=1}^t d^{i-1} \leq td^t$. Therefore, $\sum_{i=1}^{\text{diam}(G)+k} d^{i-1} \leq (\text{diam}(G) + k)d^{\text{diam}(G)+k}$. Since $n = pd^k$, $\text{diam}(G) + k = O(\log n)$ and $d^{\text{diam}(G)+k} = d^{\text{diam}(G)}d^k = O(n)$. Hence, $(\text{diam}(G) + k)d^{\text{diam}(G)+k} = O(n \log n)$. \square

From Theorems 4.2, and 4.3, a lower bound on the crossing number of $L^k(G)$ is obtained. (Note that $\sum_{i=1}^n d_i^2 = 4d^2n = O(n)$ for $U(L^k(G))$, since $L^k(G)$ is d -regular.)

Corollary 4.4 *Let G be a d -regular digraph. The crossing number of $L^k(G)$ is $\Omega(\frac{n^2}{\log^2 n})$.*

Therefore, by Theorem 4.1 and Corollary 4.4, we obtain a lower bound on the area for multilayer grid embeddings of iterated line digraphs.

Theorem 4.5 *For any fixed d -regular digraph G and for any fixed t , every t -layer grid embedding of $L^k(G)$ requires $\Omega(\frac{n^2}{\log^2 n})$ area.*

5 Specific Families of Iterated Line Digraphs

5.1 De Bruijn and Kautz Digraphs

The de Bruijn and Kautz digraphs have been noticed as interconnection networks because of their various nice properties (Bermond & Peyrat 1989, Xu 2001). The *de Bruijn digraph* $B(d, D)$ is a digraph whose vertices are the words of length D on an alphabet of d letters (for example, $\{0, 1, \dots, d-1\}$), and in which there is an arc from each vertex $(v_0, v_1, \dots, v_{D-1})$ to the d vertices $(v_1, \dots, v_{D-1}, \alpha)$, where $\alpha \in \{0, 1, \dots, d-1\}$. Thus, $B(d, D)$ is d -regular, and $|V(B(d, D))| = d^D$.

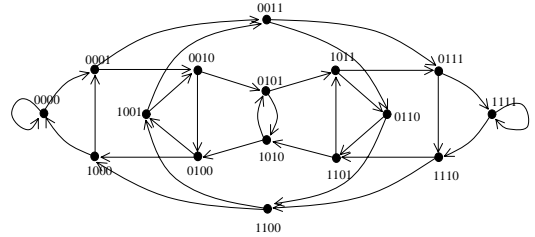


Figure 6: $B(2, 4)$.

The *Kautz digraph* $K(d, D)$ is a digraph whose vertices are the words of length D without two consecutive identical letters on an alphabet of $d + 1$ letters, and in which there is an arc from each vertex $(v_0, v_1, \dots, v_{D-1})$ to the d vertices $(v_1, \dots, v_{D-1}, \alpha)$, where $\alpha \in \{0, 1, \dots, d\}$ and $\alpha \neq v_{D-1}$. Thus, $K(d, D)$ is d -regular and $|V(K(d, D))| = d^{D-1}(d + 1)$. Figures 6 and 7 illustrate examples of de Bruijn and Kautz digraphs, respectively.

Fiol, Yebra and Alegre (1984) showed that $B(d, D)$ and $K(d, D)$ can be also defined as $B(d, D) = L^{D-1}(K_d^\circ)$ and $K(d, D) = L^{D-1}(K_{d+1}^*)$, where K_{d+1}^* is the complete symmetric digraph with $d + 1$ vertices, and K_d° is the complete digraph with d vertices, i.e., the digraph obtained from K_d^* by adding a loop to each vertex. Figure 2 also illustrates $B(2, 1)$, $B(2, 2)$ and $B(2, 3)$.

Theorem 5.1

- $B(d, D)$ can be embedded in d layers in $d^{2D}(d + 1)$ area.
- $K(d, D)$ can be embedded in d layers in $d^{2D-1}(d + 1)(d + 2)$ area.

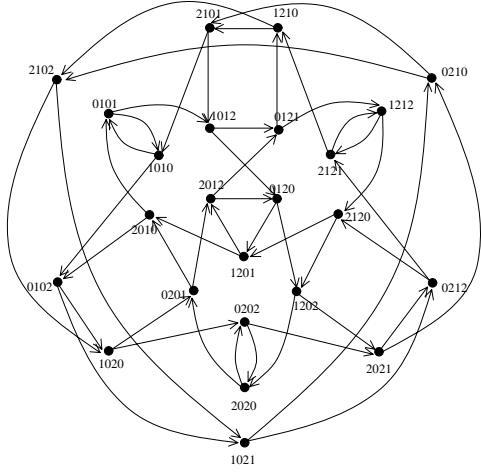


Figure 7: $K(2, 4)$.

Proof. The complete digraph K_d° is decomposed into d loop-stars. We draw each loop-star on one layer. For a d -multilayer grid embedding of K_d° , place d horizontal line segments of length d horizontally. Then, except for loops, draw each arc above the horizontal line segments. Finally, draw each loop below the horizontal line segments. (Figure 8 illustrates the 4-layer grid embedding of K_4° .) The width of this embedding is d^2 and the height of the embedding is $(d+1)$. Therefore, the area for this embedding of K_d° is $d^2(d+1)$. Since $B(d, D) = L^{D-1}(K_d^\circ)$, by applying the algorithm in Section 3, a d -multilayer grid embedding of $B(d, D)$ using $d^2(d+1) \times d^{2(D-1)} = d^{2D}(d+1)$ area is obtained.

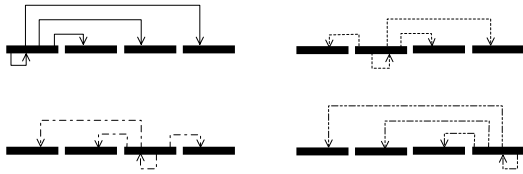


Figure 8: The 4-layer grid embedding of K_4° .

The complete symmetric digraph K_{d+1}^* is decomposed into d 2-cycle-stars such that a vertex is commonly contained in 2-cycles of all the 2-cycle-stars. Let x be such a vertex in K_{d+1}^* . Each 2-cycle-star is drawn on one layer. For d -multilayer grid embedding of K_{d+1}^* , place d horizontal line segments of length d horizontally such that $\text{lseg}(x)$ is placed in the most right position. Then, for each 2-cycle-star, draw the arcs above the horizontal line segments except for one arc in the 2-cycle. The remaining arcs are drawn below the horizontal line segments. (Figure 9 illustrates the 4-layer grid embedding of K_5^* .) The width of the embedding is $d(d+1)$ and the height of the embedding is $(d+2)$. Thus, the area for the embedding is $d(d+1)(d+2)$. Since $K(d, D) = L^{D-1}(K_{d+1}^*)$, by applying the algorithm presented in Section 3, $K(d, D)$ can be embedded in d layers in $d(d+1)(d+2) \times d^{2(D-1)} = d^{2D-1}(d+1)(d+2)$ area. \square

Using n to denote the number of vertices, Theorem

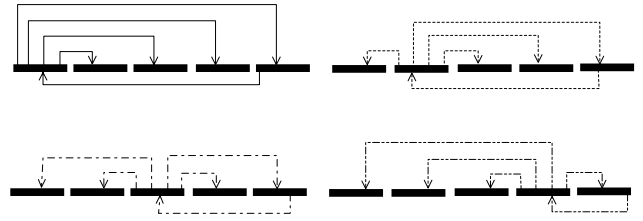


Figure 9: The 4-layer grid embedding of K_5^* .

5.1 can be restated as follows.

- $B(d, D)$ can be embedded in d layers in $(d+1)n^2$ area.
- $K(d, D)$ can be embedded in d layers in $(d+2)n^2$ area.

Remark: For a loop (v, v) in G , we can draw the paths $((v, v), (v, w))$ in $L(G)$ inside the magnified rectangular corresponding to $\text{lseg}(v)$. Thus, the area required for a d -layer grid embedding of $B(d, D)$ can be improved to dn^2 by the special treatment for loops.

5.2 Wrapped Butterfly Digraphs

The d -ary butterfly graph $b(d, r)$ is one of the most popular interconnection networks and is defined as follows:

$$V(b(d, r)) = \{(v_0, \dots, v_{r-1}, l) \mid v_i \in \{0, 1, \dots, d-1\}, \\ 0 \leq i < r, 0 \leq l \leq r\},$$

$$E(b(d, r)) = \{(u_0, \dots, u_{r-1}, l), (v_0, \dots, v_{r-1}, l+1) \mid u_i = v_i \text{ for } i \neq l\}.$$

The d -ary butterfly digraph $\vec{b}(d, r)$ is a digraph obtained from $b(d, r)$ by orienting each edge according to increasing values of l (see Figure 10).

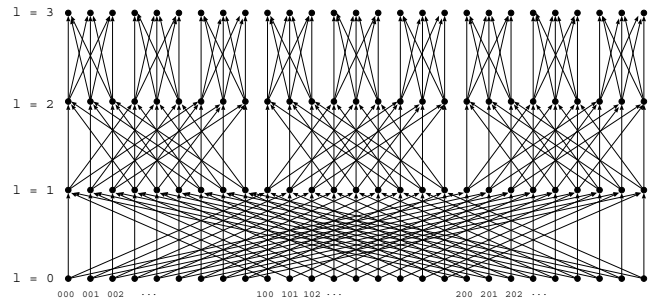


Figure 10: $\vec{b}(3, 3)$.

The d -ary wrapped butterfly graph $wb(d, r)$ is the graph obtained from $b(d, r)$ by identifying each vertex of the lowest level ($l=0$) with the corresponding vertex of the highest level ($l=r$) (Leighton 1992, Xu 2001). Similarly, the d -ary wrapped butterfly digraph $\vec{wb}(d, r)$ is obtained from $\vec{b}(d, r)$. Thus, $\vec{wb}(d, r)$ is d -regular and $|V(\vec{wb}(d, r))| = d^r r$.

Bermond *et al.* (1998) showed that $\vec{wb}(d, r)$ can be also defined as $L^{r-1}(K_d^* \otimes C_r)$, where \otimes denotes the Kronecker product and C_r is a cycle of length r . The Kronecker product of two digraphs G_1 and G_2 , denoted by $G_1 \otimes G_2$, is defined as follows:

$$\begin{aligned}
V(G_1 \otimes G_2) &= V(G_1) \times V(G_2), \\
A(G_1 \otimes G_2) &= \{((u_1, u_2), (v_1, v_2)) \mid \\
&\quad (u_1, v_1) \in A(G_1) \text{ and } (u_2, v_2) \in A(G_2)\}.
\end{aligned}$$

Theorem 5.2 $\vec{wb}(d, r)$ can be embedded in d layers in $d^{2r-1}(d+1)(dr+2)$ area.

Proof. Let $V(K_d^\circ) = \{v_1, v_2, \dots, v_d\}$. Also, let $V(C_r) = \{w_1, w_2, \dots, w_r\}$. Then $V(\vec{wb}(d, r)) = \{(v_i, w_j) \mid 1 \leq i \leq d, 1 \leq j \leq r\}$. For each j , place horizontal line segments $\text{lseg}((v_i, w_j))$ of length d , $1 \leq i \leq d$, horizontally. Also, $\text{lseg}((v_i, w_{j+1}))$, $1 \leq i \leq d$, are placed in parallel with $\text{lseg}((v_i, w_j))$, $1 \leq i \leq d$ such that the distance between them is d .

Since $K_d^* \otimes C_r$ is decomposed into d r -cycle-rooted trees, draw each r -cycle-rooted tree in one layer. For each r -cycle-rooted tree, the arcs connecting a vertex in $\{(v_i, w_j) \mid 1 \leq i \leq d\}$ and a vertex in $\{(v_i, w_{j+1}) \mid 1 \leq i \leq d\}$ can be correctly drawn in the grid space between $\{\text{lseg}((v_i, w_j)) \mid 1 \leq i \leq d\}$ and $\{\text{lseg}((v_i, w_{j+1})) \mid 1 \leq i \leq d\}$ for $j = 1, 2, \dots, r-1$. For the arcs connecting a vertex in $\{(v_i, w_r) \mid 1 \leq i \leq d\}$ and a vertex in $\{(v_i, w_1) \mid 1 \leq i \leq d\}$, $\lfloor d/2 \rfloor$ arcs are drawn in anti-clockwise manner, and $\lceil d/2 \rceil$ arcs are drawn in clockwise manner. Figure 11 illustrates such an embedding of $K_3^* \otimes C_3$.

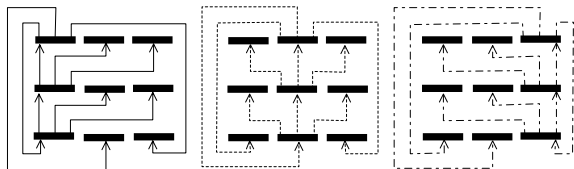


Figure 11: The 3-layer grid embedding of $K_3^* \otimes C_3$.

In this embedding of $K_d^* \otimes C_r$, the width is $d \times d + \lfloor d/2 \rfloor + \lceil d/2 \rceil = d(d+1)$, and the height is $d \times (r-1) + 1 + 2\lfloor d/2 \rfloor \leq dr+2$. Therefore, by the algorithm in Section 3, $\vec{wb}(d, r)$ can be embedded in d layers using $d(d+1)(dr+2) \times d^{2(r-1)} = d^{2r-1}(d+1)(dr+2)$ area. \square

6 Concluding remarks

We have shown that for any fixed d -regular digraph G , every iterated line digraph $L^k(G)$ ($k \geq 1$) can be embedded in d layers using $O(n^2)$ area, where n is the number of vertices in $L^k(G)$. Also, we have presented $\Omega(\frac{n^2}{\log^2 n})$ lower bound on the area of $L^k(G)$ for any fixed number of layers. Besides, we have applied the results to specific families of iterated line digraphs. It remains unknown whether our bounds can be improved.

For iterated line digraphs, several kinds of layouts have been studied (Hasunuma 2002, Hasunuma 2003). Queue layouts can be applied to three-dimensional straight-line grid drawings. Book-embeddings (also called stack-layouts) are related to multilayer grid embeddings. A book consists of a line called the *spine* and half-planes sharing the spine as a common boundary, called *pages*. A book-embedding of a graph is defined by a linear ordering of the vertices, and an assignment of the edges to pages so that there is no crossing of edges in each page. It is not difficult to

see that a $2d$ -page book-embedding of a graph can be applied to obtain a d -layer grid embedding of a graph by considering two pages as one layer.

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