Minimum Augmentation of Edge-Connectivity with Monotone Requirements in Undirected Graphs

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Abstract

For a finite ground set V, we call a set-function $r : 2^V \to Z^+$ monotone, if $r(X') \geq r(X)$ holds for each $X' \subseteq X \subseteq V$, where Z^+ is the set of nonnegative integers. Given an undirected multigraph $G = (V, E)$ and a monotone requirement function $r : 2^V \to Z^+$, we consider the problem of augmenting G by a smallest number of new edges so that the resulting graph G' satisfies $d_{G'}(X) \geq r(X)$ for each $\emptyset \neq X \subset V$, where $d_G(X)$ denotes the degree of a vertex set X in G. This problem includes the edge-connectivity augmentation problem, and in general, it is NP-hard even if a polynomial time oracle for r is available. In this paper, we show that the problem can be solved in $O(n^4(m + n \log n + q))$ time, under the assumption that each $\emptyset \neq X \subset V$ satisfies $r(X) \geq 2$ whenever $r(X) > 0$, where $n = |V|$, $m = |\{\{u, v\} \mid (u, v) \in E\}|$, and q is the time required to compute $r(X)$ for each $X \subseteq V$.

Keywords: undirected graph, edge-connectivity, connectivity augmentation problem, monotone requirement, polynomial time deterministic algorithm

1 Introduction

In a communication network, graph connectivity is a fundamental measure of its robustness. The connectivity augmentation problems have been extensively studied as an important subject in the network design problem (Grötschel, Monma & Stoer 1995) and so on, and many efficient algorithms have been developed so far (see (Frank 1994, Nagamochi & Ibaraki 2002) for surveys).

Let $G = (V, E)$ be an undirected multigraph and $d_G(X)$ be the number of edges between X and $V - X$ in G. A graph $G = (V, E)$ is k-edge-connected if every set $\emptyset \neq X \subset V$ satisfies $d_G(X) \geq k$. We consider the following problem of augmenting a given graph to meet the required edge-connectivity (RECAP): given a graph $G = (V, E)$ and a nonnegative integer set-function $r : 2^V \to Z^+$ where Z^+ denotes the set of nonnegative integers, add a smallest number of new edges F so that the augmented graph $G + F = (V, E \cup F)$ satisfies $d_{G+F}(X) \geq r(X)$ for every $\emptyset \neq X \subset V$. This formulation includes the *edge-connectivity augmentation problem* (*ECAP*) , the *local edge-connectivity augmentation problem* (*LECAP*), the *node-to-area edge-connectivity augmentation problem* (*NAECAP*), and so on.

Let us briefly survey the developments in the edge-connectivity augmentation problems. ECAP is equivalent to RECAP in the case where every $\emptyset \neq X \subset V$ satisfies $r(X) = k$ for a given integer $k \in Z^+$. Watanabe and Nakamura (1987) showed that it is polynomially solvable. The fastest known algorithm for it achieves complexity $O(mn + n^2 \log n)$ due to Nagamochi (2003), where $n = |V|$ and $m = |\{\{u, v\} \mid u, v \in V\}|$.

In LECAP, we are given a local edge-connectivity requirement function $r'(u, v) \in Z^+$ on the set of pairs of vertices u and v, and hence the function r in RECAP is regarded as $r(X) = \max\{r'(u, v) \mid u \in X, v \in V - X\}$. Clearly, LECAP includes ECAP as a special case. Frank (1992) showed that it is polynomially solvable. The fastest known algorithm, proposed by Gabow (1994), runs in $O(n^2 m \log(n^2/m))$ time.

In NAECAP, we are given a family $\mathcal{W}$ of specified vertex subsets called *areas* and a requirement function $r'(W)$ on the family of areas $W \in \mathcal{W}$, and asked to augment G so that the edge-connectivity between each pair of $W \in \mathcal{W}$ and $v \in V - W$ becomes at least $r'(W)$; in the augmented graph G', every set $\emptyset \neq X \subset V$ is required to satisfy $d_{G'}(X) \geq r'(W)$ for each area $W \in \mathcal{W}$ with $W \cap X = \emptyset$ or $W \subseteq X$. Hence, the function r in RECAP is regarded as $r(X) = \max\{r'(W) \mid W \cap X = \emptyset$, or $W \subseteq X\}$. NAECAP is also an extension of ECAP, because if $r'(W) = k$ holds for each area $W \in \mathcal{W}$ and some area $W' \in \mathcal{W}$ satisfies $|W'| = 1$, then the function r satisfies $r(X) = k$. Miwa and Ito (2004) showed that even if $r'(W) = 1$ holds for every area $W \in \mathcal{W}$, NAECAP is NP-hard. On the other hand, Ishii and Hagiwara (2006) showed that the case where $r'(W) \geq 2$ for every area $W \in \mathcal{W}$ can be solved in $O(n^3|\mathcal{W}|(m + n \log n))$ time.

More generally, RECAP can be extended to a problem of *covering* a given nonnegative integer set-function $p : 2^V \to Z^+$ by a smallest number of graph edges, where we say that an edge set F covers p if $d_{(V,F)}(X) \geq p(X)$ for every $X \subseteq V$. The p in RECAP is regarded as $p(X) = \max\{0, \max\{r(X), r(V - X)\} - d_G(X)\}$ (note that the degree of each set $\emptyset \neq X \subset V$ needs to be augmented up to $\max\{r(X), r(V - X)\}$ since G is undirected). Benczúr and Frank (1999) showed that if p is a *symmetric supermodular* set-function, then such a problem of covering p can be solved in polynomial time, where $p : 2^V \to Z^+$ is symmetric if $p(X) = p(V - X)$ for every $X \subseteq V$, and p is (crossing) supermodular if $p(\emptyset) = 0$ and

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \qquad (1.1)$$

for every $X, Y \subseteq V$ with $p(X) > 0$, $p(Y) > 0$ and $X \cap Y \neq \emptyset \neq V - (X \cup Y)$. Since $-d_G$ is symmetric supermodular, ECAP is a special case of this problem.

On the other hand, the functions p defined in LECAP and NAECAP are not symmetric supermodular, but symmetric *skew-supermodular*, as observed in (Frank 1992) and (Ishii & Hagiwara 2006), respectively, where $p : 2^V \to Z^+$ is skew-supermodular if $p(\emptyset) = 0$, and at least one of (1.1) and

$$p(X) + p(Y) \le p(X - Y) + p(Y - X) \qquad (1.2)$$

hold for every $X, Y \subseteq V$ with $p(X) > 0$ and $p(Y) > 0$. Note that the problem of covering symmetric skew-supermodular functions is NP-hard since so is NAECAP. Recently, Nutov (2005) proved that this problem is APX-hard and 7/4-approximable in polynomial time under the assumption that a polynomial time oracle for $\min_{X\subseteq V}\{\sum_{v\in X} g(v) + d_{(V,F)}(X) - p(X)\}$ is available, where $g : V \to Z^+$ is a function on V and F denotes a set of edges on V (note that such an oracle for a supermodular function p is always available as pointed in (Benczúr & Frank 1999)). Some other problems as the *element-connectivity augmentation problem* (*ELCAP*) are also included in this problem as its special case, and ELCAP was shown to be NP-hard even if $r \in \{0, 2\}$ (Király, Cosh & Jackson 1999, Nutov 2005). It remains a challenging question which type of the problem of covering symmetric skew-supermodular functions is polynomially solvable or not.

In this paper, we consider the *edge-connectivity augmentation problem with monotone requirements* (*MECAP*), which is RECAP with a *monotone* function r, where $r : 2^V \to Z^+$ is monotone if $r(X') \ge r(X)$ holds for every two sets $X', X \subseteq V$ with $\emptyset \ne X' \subseteq X$. NAECAP with $\mathcal{W}$ and $r' : \mathcal{W} \to Z^+$ is equivalent to MECAP with r'', where $r''(X) = \max\{r'(W) \mid W \cap X = \emptyset\}$ for each $\emptyset \ne X \subset V$. Indeed, the function r'' is monotone and the function r in NAECAP satisfies $r(X) = \max\{r''(X), r''(V - X)\}$. On the other hand, MECAP with r is equivalent to NAECAP with $\mathcal{W} = \{W \subset V \mid r(V-W) > 0\}$ and $r'(W) = r(V-W)$, $W \in \mathcal{W}$. Indeed, for each $\emptyset \ne X \subset V$, we have $\max\{r'(W) \mid W \cap X = \emptyset, W \in \mathcal{W}\} = r(X)$ by the monotonicity of r. In this sense, we may say that MECAP is a reformulation of NAECAP. It follows that the function p defined in MECAP is symmetric skew-supermodular and MECAP is NP-hard in general. However, the method of applying Ishii and Hagiwara's algorithm (2006) to NAECAP with $\mathcal{W} = \{W \subset V \mid r(V - W) > 0\}$ and $r'(W) = r(V - W)$, $W \in \mathcal{W}$ is not a polynomial time one for MECAP, because their algorithm depends on the number of areas and $|\{W \subset V \mid r(V - W) > 0\}|$ may be exponential in n and m. In this paper, we propose an algorithm for solving MECAP in $O(n^4(m + n \log n + q))$ time, under the assumption that each $\emptyset \ne X \subset V$ satisfies $r(X) \ge 2$ whenever $r(X) > 0$, where q is the time required to compute $r(X)$ for each $X \subseteq V$; this gives rise to a polynomial time algorithm under the assumption that q is polynomial in the input size of the problem. In NAECAP with $\mathcal{W}$ and r', we have $r(X) = \max\{r'(W) \mid W \cap X = \emptyset\}$, and hence $r(X)$ can be computed in $O(\sum_{W\in\mathcal{W}} |W|)$ time; our algorithm is a polynomial time one also for NAECAP under the assumption that $r'(W) \ge 2$ holds for each $W \in \mathcal{W}$. Moreover, its time complexity improves Ishii and Hagiwara's one (Ishii & Hagiwara 2006) in some case; e.g., in the case of $n = o(|\mathcal{W}|)$ and $\sum_{W\in\mathcal{W}} |W| = O(m + n \log n)$.

The paper is organized as follows. In Section 2, we define MECAP, after introducing some basic notations. In Section 3, we derive lower bounds on the optimal value to MECAP, and state our main result that MECAP is polynomially solvable under the assumption that $r(X) \ge 2$ holds for every $X \subseteq V$ whenever $r(X) > 0$. In Section 4, we introduce the so-called edge-splitting operation, and give an algorithm for solving MECAP, based on these lower bounds and the edge-splitting operation. In Section 5, we prove the correctness of the algorithm. In Section 6, we give concluding remarks. Due to space limitation, some proofs are omitted.

2 Problem Definition

Let $G = (V, E)$ stand for an undirected graph with a set V of *vertices* and a set E of *edges*. An edge with end vertices u and v is denoted by (u, v). We denote $|V|$ by n (or by $n(G)$) and $|\{\{u, v\}|(u, v) \in E\}|$ by m (or by $m(G)$). A singleton set $\{x\}$ may be simply written as x, and " $\subset$ " implies proper inclusion while " $\subseteq$ " means " $\subset$ " or " $=$ ". In $G = (V, E)$, its vertex set V and edge set E may be denoted by $V(G)$ and $E(G)$, respectively. A maximal connected subgraph G' in a graph G is called a *component* of G (for notational convenience, a component H may be represented by its vertex set $X = V(H)$). For a subset $V' \subseteq V$ in G, the subgraph induced by V' is denoted by $G[V']$ or $G - (V - V')$. For an edge set E' with $E' \cap E = \emptyset$, we denote the augmented graph $(V, E \cup E')$ by $G + E'$. For an edge set E', we denote by $V[E']$ the set of all end vertices of edges in E'.

For two disjoint subsets $X, Y \subset V$ of vertices, we denote by $E_G(X, Y)$ the set of edges $e = (x, y)$ such that $x \in X$ and $y \in Y$, and also denote $|E_G(X, Y)|$ by $d_G(X, Y)$. In particular, $d_G(X, V - X)$ may be written as $d_G(X)$. Moreover, we define $d_G(\emptyset) = d_G(V) = 0$. For two sets $X, Y \subseteq V$ in a graph $G = (V, E)$, we say that X and Y *intersect* each other in G if none of $X \cap Y$, $X - Y$, $Y - X$ is empty. For a graph $G = (V, E)$, every two sets $X, Y \subset V$ satisfy the following equalities.

$$\begin{aligned} d_G(X) + d_G(Y) &= d_G(X \cap Y) + d_G(X \cup Y) \\ &\quad + 2d_G(X - Y, Y - X). \qquad (2.1) \\ d_G(X) + d_G(Y) &= d_G(X - Y) + d_G(Y - X) \\ &\quad + 2d_G(X \cap Y, V - (X \cup Y)) (2.2) \end{aligned}$$

Given a ground set V, a set-function $r : 2^V \to Z^+$ is called *monotone* if $r(X') \ge r(X)$ holds for each set X, X' with $\emptyset \ne X' \subseteq X \subseteq V$. In this paper, we consider the following connectivity augmentation problem with monotone requirements.

Problem 2.1 (Edge-connectivity augmentation problem with monotone requirements, MECAP)
Input: *An undirected graph $G = (V, E)$ and a monotone function $r : 2^V \to Z^+$.*
Output: *A set E^* of new edges with the minimum cardinality such that each set $\emptyset \ne X \subset V$ satisfies $d_{G+E^*}(X) \ge r(X)$.* □

We call a set $X \subseteq V$ *r-maximal* if $r(X) > 0$ and $r(X') = 0$ holds for each set $X' \supset X$. Let $\mathcal{R}$ denote the family of r-maximal subsets of V. A set $\emptyset \ne X \subset V$ is called *proper* if $X \subseteq M$ or $V - X \subseteq M$ for some $M \in \mathcal{R}$. Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the family of proper sets X such that X (resp. $V - X$) is contained in some r-maximal set (note that some proper set may belong to both of $\mathcal{A}$ and $\mathcal{B}$). From the symmetry of d_G, a set F of edges is feasible to MECAP if and only if all proper sets X satisfy $d_{G+F}(X) \ge R(X)$, where $R(X) = \max\{r(X), r(V - X)\}$. For a set-function $p' : 2^V \to Z^+$, we say that an edge set E' *covers* p' if $d_{(V,E')}(X) \ge p'(X)$ for each set $X \subseteq V$. We remark that a set E' of edges is feasible to MECAP if and only if E' covers p, where

$$p(X) = \max\{0, R(X) - d_G(X)\} \text{ for every set } \emptyset \ne X \subset V, \text{ and } p(\emptyset) = p(V) = 0.$$

As mentioned in Section 1, p is symmetric skew-supermodular. We here give its proof for completing the paper.

Lemma 2.2 *Let $r : 2^V \to Z^+$ be a monotone set-function on V. Then p is symmetric skew-supermodular.* □

Let $\mathcal{A}^*$ (resp. $\mathcal{B}^*$) denote the family of proper sets X in $\mathcal{A}$ (resp. $\mathcal{B}$) with $r(X) \geq r(V-X)$ (resp. $r(X) \leq r(V-X)$). Note that each proper set belongs to $\mathcal{A}^*$ or $\mathcal{B}^*$ and that $X \in \mathcal{A}^*$ if and only if $V - X \in \mathcal{B}^*$. By the monotonicity of r, it is not difficult to see the following.

Lemma 2.3 *Let $r : 2^V \to Z^+$ be a monotone set-function on V and X be a proper subset in $G = (V, E)$.*
(i) *If $X \in \mathcal{A}^*$, then any set $\emptyset \neq X' \subseteq X$ belongs to $\mathcal{A}$ and $R(X') \geq r(X') \geq R(X)$.*
(ii) *If $X \in \mathcal{B}^*$, then any set $V \supset X' \supseteq X$ belongs to $\mathcal{B}$ and $R(X') \geq r(V - X') \geq R(X)$.* □

PROOF of Lemma 2.2: Clearly, p is symmetric by the symmetry of d_G and R. Since d_G satisfies both of (2.1) and (2.2), it suffices to show that R is skew-supermodular. For this, we show that every two intersecting proper subsets X, Y of V with $p(X), p(Y) > 0$ satisfy the followings (note that the cases of $X \subseteq Y$ or $X \cap Y = \emptyset$ clearly satisfy (1.1) or (1.2)):

If (a) $X, Y \in \mathcal{A}^*$, (b) $X, Y \in \mathcal{B}^*$, or (c) $X \in \mathcal{A}^*$, $Y \in \mathcal{B}^*$, and $V = X \cup Y$, then $R(X) + R(Y) \leq R(X - Y) + R(Y - X)$. (2.3)

If $X \in \mathcal{A}^*$, $Y \in \mathcal{B}^*$, $V \neq X \cup Y$, then $R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y)$. (2.4)

In the case of (a) (resp. (b)), Lemma 2.3(i) implies that $R(X-Y) \geq r(X-Y) \geq R(X)$ and $R(Y-X) \geq r(Y-X) \geq R(Y)$ (resp. $R(Y-X) \geq r(Y-X) \geq R(V-X) = R(X)$ and $R(X-Y) \geq r(X-Y) \geq R(V-Y) = R(Y)$ from $V-X, V-Y \in \mathcal{A}^*$), implying (2.3). In the case of (c), $R(X-Y) = R(V-Y) = R(Y)$ and $R(Y-X) = R(V-X) = R(X)$ imply (2.3). In the remaining case, we have $R(X \cap Y) \geq R(X)$ (resp. $R(X \cup Y) \geq R(Y)$) by Lemma 2.3(i) (resp. by Lemma 2.3(ii) and $V \neq X \cup Y$), which implies (2.4). □

3 Lower Bound on the Optimal Value

For a graph G and a fixed function $r : 2^V \to Z^+$, let $opt(G, r)$ denote the optimal value to MECAP in G, i.e., the minimum size $|E^*|$ of a set E^* of new edges which covers p. In this section, we derive lower bounds on $opt(G, r)$ to MECAP with G and r.

A family $\mathcal{X} = \{X_1, \ldots, X_t\}$ of nonempty vertex sets in $G = (V, E)$ is called a *subpartition of* V, if every two sets $X_i, X_j \in \mathcal{X}$ satisfy $X_i \cap X_j = \emptyset$. If X is proper, then it is necessary to add at least $p(X)$ edges between X and $V - X$. Let

$$\alpha(G, r) = \max_{\mathcal{X}} \left\{ \sum_{X \in \mathcal{X}} p(X) \right\}, \qquad (3.1)$$

where the maximization is taken over all subpartitions of V. Then any feasible solution to MECAP with G and r must contain an edge which joins two vertices from a set X with $p(X) > 0$ and the set $V - X$. Therefore we see the following lemma.

Remark 3.1 *$opt(G, r) \geq \lceil \alpha(G, r)/2 \rceil$ holds.* □

We remark that there is an instance with $opt(G, r) > \lceil \alpha(G, r)/2 \rceil$. Figure 1 gives an instance where $\mathcal{R} = \{M_1, M_2, M_3\}$ and all proper sets X satisfies $R(X) = 2$. Each set $\{v_i\}$, $i = 1, 2, 3, 4, 5$ is proper, $p(v_i) = 2 - d_G(v_i) = 1$ for $i = 1, 2, 3, 5$ and $p(v_4) = 2 - d_G(v_4) = 2$. It is not hard to see that in (3.1) the maximum is achieved for the subpartition $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$ and $\lceil \alpha(G, r)/2 \rceil = 3$. In order to obtain a feasible solution of three edges, we must add $E' = \{(v_1, v_2), (v_3, v_4), (v_4, v_5)\}$ or $E' = \{(v_1, v_4), (v_2, v_4), (v_3, v_5)\}$ without loss of generality. In both cases, E' is infeasible because the proper set X satisfies $d_{G+E'}(X) = 1$ for $X = M_1 - \{v_4, v_5\}$ in the former case and $X = M_1 - \{v_5\}$ in the latter case. We will show that all such instances can be completely characterized.

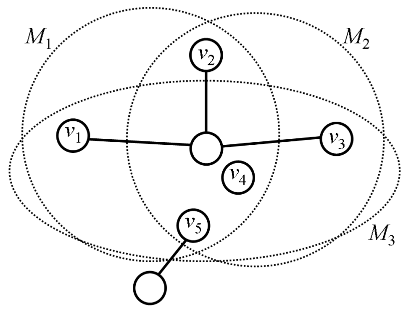

Figure 1: Illustration of a graph G with $opt(G, r) > \lceil \frac{\alpha(G,r)}{2} \rceil$.

Definition 3.2 *We say that a graph G has* property (P) *if there is a subpartition $\mathcal{X}$ of V with $\sum_{X \in \mathcal{X}} p(X) = \alpha(G, r)$ satisfying the following conditions (P1)–(P3) :*
(P1) $\alpha(G, r)$ is even.
(P2) There is a set $X^* \in \mathcal{X}$ with $p(X^*) = 1$.
(P3) Let $\mathcal{X}_1$ denote the family of proper sets $X \in \mathcal{X}$ with $d_G(X) = 0$ and $p(X) = 2$. For each $X \in \mathcal{X} - \mathcal{X}_1 - \{X^*\}$, there is a set $Y_X \in \mathcal{B}^*$ such that the following (i)–(iv) hold: (i) $X \cup X^* \subseteq Y_X$, (ii) $V - Y_X - (\cup_{X'' \in \mathcal{X}_1} X'') \neq \emptyset$, (iii) $\sum_{X' \in \mathcal{X}, X' \subset Y_X} p(X') \leq p(Y_X) + 1$, and (iv) every set $X' \in \mathcal{X}$ satisfies $X' \subset Y_X$ or $X' \cap Y_X = \emptyset$. □

Note that G in Figure 1 has property (P) because $\alpha(G, r) = 6$ holds and the subpartition $\mathcal{X} = \{X^* = \{v_5\}, X_1 = \{v_1\}, X_2 = \{v_2\}, X_3 = \{v_3\}, X_4 = \{v_4\}\}$ of V satisfies $\mathcal{X}_1 = \{X_4\}$, $Y_{X_1} = (V - M_2) \cup \{v_5\}$, $Y_{X_2} = (V - M_3) \cup \{v_5\}$, and $Y_{X_3} = (V - M_1) \cup \{v_5\}$.

Lemma 3.3 *If G has property* (P)*, then $opt(G, r) \geq \lceil \alpha(G, r)/2 \rceil + 1$.*

Proof: Assume by contradiction that G has property (P) and there is an edge set E^* with $|E^*| = \alpha(G, r)/2$ such that E^* covers p (note that $\alpha(G, r)$ is even by the property (P1)). Let $\mathcal{X} = \{X_1, \ldots, X_t\}$ denote a subpartition of V satisfying $\sum_{X \in \mathcal{X}} p(X) = \alpha(G, r)$, $p(X) > 0$ for each $X \in \mathcal{X}$, and the above (P2) and (P3). Since $|E^*| = \alpha(G, r)/2$ holds, each set $X \in \mathcal{X}$ satisfies $d_{G'}(X) = p(X)$, where $G' = (V, E^*)$. Therefore, any edge $(x, x') \in E^*$ satisfies $x \in X$ and $x' \in X'$ for some two sets $X, X' \in \mathcal{X}$ with $X \neq X'$. Hence $\sum_{v \in X''} d_{G'}(v) = d_{G'}(X'')$ for $X'' \in \{X, X'\}$. From this, there exists a set $X_1 \in \mathcal{X} - \{X^*\}$ with $E_{G'}(X^*, X_1) \neq \emptyset$. Now note that $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$

holds since otherwise $\alpha(G,r) = 2|\mathcal{X}_1|+1$ by the properties (P2) and (P3), contradicting that $\alpha(G,r)$ is even.

Assume that $X_1 \in \mathcal{X} - \mathcal{X}_1$ holds. Since G satisfies property (P), there is a set $Y_{X_1} \in \mathcal{B}^*$ which satisfies (P3), and hence $\sum_{v\in Y_{X_1}} d_{G'}(v) = \sum_{X'\in\mathcal{X},X'\subset Y_{X_1}} d_{G'}(X') = \sum_{X'\in\mathcal{X},X'\subset Y_{X_1}} p(X') \leq p(Y_{X_1}) + 1$. Since $G'[Y_{X_1}]$ contains one edge in $E_{G'}(X_1, X^*)$, the proper set Y_{X_1} satisfies $d_{G'}(Y_{X_1}) \leq p(Y_{X_1}) - 1$, which contradicts that E^* covers p.

Assume that $X_1 \in \mathcal{X}_1$. From the properties (P2) and (P3), we have $d_{G'}(X^* \cup X_1) = 1$, and this implies that there exists an edge $e \in E^*$ connecting X_1 and some set in $\mathcal{X} - \{X^*, X_1\}$. Let $\mathcal{X}'_1 = \{X^*, X_1, X_2, \ldots, X_{t'}, X_{t'+1}\}$ be the family of sets in $\mathcal{X}$ such that we have $X_i \in \mathcal{X}_1$ for each $i = 1, 2, \ldots, t'$ and $X_{t'+1} \in \mathcal{X} - \mathcal{X}_1$ and $E_{G'}(X_i, X_{i+1}) \neq \emptyset$ for each $i = 1, \ldots, t'$ (note that such $X_{t'+1}$ exists by $\mathcal{X} - \mathcal{X}_1 - \{X^*\} \neq \emptyset$). Note that such $\mathcal{X}'_1$ is determined uniquely by

$$d_{G'}(X^*) = 1 \text{ and } d_{G'}(X) = 2 \text{ for each } X \in \mathcal{X}_1. \tag{3.2}$$

From the definition of property (P), there is a set $Y_{X_{t'+1}} \in \mathcal{B}^*$ satisfying (P3) for $X_{t'+1}$. Let $Y_{t'+1} = Y_{X_{t'+1}} \cup (\cup_{X\in\mathcal{X}'_1} X)$. Since we have $Y_{t'+1} \supseteq Y_{X_{t'+1}} \in \mathcal{B}^*$ and $V - Y_{X_{t'+1}} - (\cup_{X\in\mathcal{X}_1} X) \neq \emptyset$ (by the property (P3)), Lemma 2.3(ii) implies that $Y_{t'+1}$ is also proper and $R(Y_{t'+1}) \geq R(Y_{X_{t'+1}})$. Note that $d_G(Y_{t'+1}) = d_G(Y_{X_{t'+1}})$ by $d_G(X) = 0$ for each $X \in \mathcal{X}_1$. It follows that $p(Y_{t'+1}) \geq p(Y_{X_{t'+1}})$. Thus, we have

$$\sum_{v\in Y_{t'+1}} d_{G'}(v) \leq (p(Y_{t'+1}) + 1) + 2t' \tag{3.3}$$

by $\sum_{v\in Y_{X_{t'+1}}} d_{G'}(v) \leq p(Y_{X_{t'+1}}) + 1$, (3.2), and $p(Y_{t'+1}) \geq p(Y_{X_{t'+1}})$. Also by (3.2), we can observe that each edge in E^* incident to $(\cup_{X\in\mathcal{X}'_1-\{X_{t'+1}\}} X)$ is contained in $E(G'[Y_{t'+1}])$; $E(G'[Y_{t'+1}])$ contains at least $t' + 1$ edges in E^*. From (3.3) and this, we have $d_{G'}(Y_{t'+1}) \leq (p(Y_{t'+1}) + 1) + 2t' - 2(t' + 1) = p(Y_{t'+1}) - 1$. Thus this contradicts that E^* covers p. □

In this paper, we prove that MECAP enjoys the following min-max theorem.

Theorem 3.4 *Let $G = (V,E)$ be an undirected graph and $r : 2^V \to Z^+$ be a monotone set-function on V such that $r(X) \geq 2$ holds whenever $r(X) > 0$. Then, for MECAP, $opt(G,r) = \lceil\alpha(G, r)/2\rceil$ holds if G does not have property (P), and $opt(G,r) = \lceil\alpha(G, r)/2\rceil + 1$ holds otherwise. Moreover, a solution E^* with $|E^*| = opt(G,r)$ can be obtained in $O(n^4(m + n\log n + q))$ time.* □

4 Edge-Splittings and Algorithm

4.1 Extensions

We adapt the so-called "edge-splitting" method for solving MECAP, which is known to be useful for solving connectivity augmentation problems (Frank 1992). In the edge-splitting method, after creating a new vertex s outside of G and adding new edges between s and G, we find an appropriate edge set to be added to G by splitting off a pair of edges incident to s in the extended graph. Given a graph $G = (V,E)$ and a function $r : 2^V \to Z^+$ on V, a graph $H = (V \cup \{s\}, E \cup F)$ obtained from G by adding a new vertex s and a set F of new edges connecting s and V is called a *p-extension of* G if

$$\text{all sets } X \subseteq V \text{ satisfy } d_H(s,X) \geq p(X). \tag{4.1}$$

In particular, a p-extension $H = (V \cup \{s\}, E \cup F)$ of G is called *critical* if $(V \cup \{s\}, E \cup F')$ violates (4.1) for any $F' \subset F$. In (Frank 1992, Nutov 2005), it was shown that if p is symmetric skew-supermodular, then any critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G satisfies $|F| = \alpha(G,r)$. From this and Lemma 2.2, we have the following theorem.

Theorem 4.1 *Let $G = (V,E)$ be a graph and $r : 2^V \to Z^+$ be a monotone function on V. Any critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G satisfies $|F| = \alpha(G,r)$.* □

4.2 Edge-splitting theorems

For a graph $H = (V \cup \{s\}, E)$ and a designated vertex $s \notin V$, an operation called *edge-splitting* (*at* s) is defined as deleting two edges $(s,u), (s,v) \in E$ and adding one new edge (u,v). That is, the graph $H' = (V \cup \{s\}, (E - \{(s,u),(s,v)\}) \cup \{(u,v)\})$ is obtained from such edge-splitting operation. Then we say that H' is obtained from H by *splitting* a pair of edges (s,u) and (s,v) (or by splitting (s,u) and (s,v)). A sequence of splittings is *complete* if the resulting graph H' does not have any neighbor of s.

Given a p-extension $H = (V \cup \{s\}, E \cup F)$ of $G = (V,E)$, a pair $\{(s,u),(s,v)\}$ is called *admissible* if the graph H' obtained from H by splitting (s,u) and (s,v) is also a p^{uv}-extension of $H' - s = G + \{(u,v)\}$, where $p^{uv}(X) = \max\{0, p(X) - 1\}$ for each set X with $|\{u,v\} \cap X| = 1$ and $p^{uv}(X) = p(X)$ otherwise. Notice that given a graph G, if there is a complete admissible splitting at s in its critical p-extension $H = (V \cup \{s\}, E \cup F)$, then the set E' of split edges is an optimal solution of MECAP to G and r. Indeed, in $H' = (V \cup \{s\}, E \cup E')$, $d_{H'}(s) = 0$ holds, and every set $\emptyset \neq X \subset V$ satisfies $0 = d_{H'}(s,X) \geq \max\{0, R(X) - d_{G+E'}(X)\}$, implying that E' is feasible to MECAP. Moreover, Theorem 4.1 implies that $|E'| = |F|/2 = \lceil\alpha(G,r)/2\rceil$, which is a lower bound on *opt*(G,r) by Remark 3.1. However, as indicated by Lemma 3.3, any critical p-extension of G with property (P) does not have a complete admissible splitting. If

$$\text{every set } X \subseteq V \text{ satisfies } r(X) \geq 2 \text{ whenever } r(X) > 0, \tag{4.2}$$

then we can characterize a graph with property (P) as follows.

Definition 4.2 *A p-extension $H = (V \cup \{s\}, E \cup F)$ of G has* property *(P^*) if H is a critical p-extension of G satisfying the following ($P1^*$) − ($P4^*$) :*

($P1^*$) $d_H(s)$ is even.
($P2^*$) G has exactly one component $C^* \subseteq V$ with $d_H(s,C^*) = 1$.
($P3^*$) For the edge (s,u^*) with $\{(s,u^*)\} = E_H(s,C^*)$, u^* is contained in a proper set $X \subseteq C^*$ with $d_H(s,X) = p(X)$.
($P4^*$) Let $\mathcal{C}_1$ be the family of all components C of G such that $d_H(C) = d_H(s,C) = 2$ and C is proper. For any edge $e \in E_H(s, V - \cup_{C\in\mathcal{C}_1} C)$, $\{(s,u^*), e\}$ is not admissible in H. □

Theorem 4.3 *Let $G = (V,E)$ be a graph and $r : 2^V \to Z^+$ be a monotone function satisfying (4.2). Then, G has property (P) if and only if its critical p-extension has property (P^*).* □

Moreover, the following properties hold about admissible splittings.

Theorem 4.4 *Let $r : 2^V \to Z^+$ be a monotone function on V satisfying (4.2) and $H = (V \cup \{s\}, E \cup F)$ be a critical p-extension of G. Then the following (i) and (ii) hold:*
(i) Some graph H' obtained from H by adding at most one extra edge to G and some one extra edge incident to s to make the degree of s even (if necessary) has a complete admissible splitting at s.
(ii) If H does not have property (P^*), then H has a complete admissible splitting at s after replacing at most one edge incident to s with some new edge incident to s, and adding some one extra edge incident to s to make the degree of s even (if necessary). □

We give proofs of these two theorems in Section 5. Note that Lemma 3.3, Theorem 4.1, and Theorem 4.4(ii) prove the necessity of Theorem 4.3. Indeed, if a critical p-extension H of G does not have property (P^*), then by a complete admissible splitting according to Theorem 4.4(ii), we can obtain a feasible solution E' to MECAP with G and r such that $|E'| = \lceil d_H(s)/2 \rceil = \lceil \alpha(G, r)/2 \rceil$ (by Theorem 4.1), from which and Lemma 3.3 it follows that G does not have property (P). Let us discuss its consequences. Based on these two theorems, we give the following algorithm which delivers an optimal solution to MECAP with G and r satisfying (4.2).

Algorithm M-AUG
Input: A graph $G = (V, E)$ and a monotone function $r : 2^V \to Z^+$ on V satisfying (4.2).
Output: A set E^* of new edges with $|E^*| = opt(G, r)$ which covers p.
Step 1: Find a critical p-extension $H = (V \cup \{s\}, E \cup F)$ of G.
Step 2: If H does not have property (P*), then find a complete admissible splitting at s after replacing some one edge incident to s and adding some one edge between s and V to make the degree of s even according to Theorem 4.4(ii). Otherwise, after adding some edge to G according to Theorem 4.4(i), find a complete admissible splitting at s. Output the set E^* of all edges added to G as an optimal solution. □

The details for Step 2 and the analysis of the time complexity of the algorithm will be given in Section 5. We here only observe that the set E^* obtained by the algorithm is optimal. If H does not have property (P*), then as observed above, we have $|E^*| = \lceil \alpha(G, r)/2 \rceil$, which is equal to a lower bound on $opt(G, r)$ by Remark 3.1. If H have property (P*), then $|E^*| = \lceil \alpha(G, r)/2 \rceil + 1$. Theorem 4.3 and Lemma 3.3 imply that also in this case, $|E^*|$ is equal to a lower bound on $opt(G, r)$.

5 Correctness of algorithm M-AUG

For proving the correctness of algorithm M-AUG, we have to show Theorems 4.3 and 4.4. After showing several properties about admissible splittings, we first show Theorem 4.4 in Section 5.1, which also proves the necessity of Theorem 4.3. In Section 5.2, we prove the sufficiency of Theorem 4.3; we give a proof that if a p-extension of G satisfies property (P*), then G has property (P).

Through this section, for a p-extension H of $G = (V, E)$, let $\mathcal{C}_1$ be the family of all components C of G such that $d_H(C) = d_H(s, C) = 2$ and C is proper, and $V_1 = \cup_{C \in \mathcal{C}_1} C$. Let $\mathcal{C}_2$ be the family of all components C of G such that $C \notin \mathcal{C}_1$ and $d_H(s, C) > 0$, and $V_2 = \cup_{C \in \mathcal{C}_2} C$.

We first show preparatory properties for proving the theorems. For seeking admissible pairs, we need to analyze situations where some splitting fails. For a p-extension $H = (V \cup \{s\}, E \cup F)$ of $G = (V, E)$, a pair $\{(s, u), (s, v)\} \subseteq F$ of two edges is not admissible if there is a proper set $Y \subset V$ with $\{u, v\} \subseteq Y$ and $d_H(s, Y) - p(Y) \leq 1$ (note that the graph H' obtained from H by splitting (s, u) and (s, v) satisfies $d_{H'}(s, Y) = d_H(s, Y) - 2 \leq p(Y) - 1 = p^{uv}(Y) - 1$). Also note that $d_H(s, Y) \geq 2$ implies that $p(Y) \geq d_H(s, Y) - 1 > 0$. Such set Y is called a *dangerous set*. Conversely, a pair $\{(s, u), (s, v)\}$ is not admissible only if there is a dangerous set $Y \subset V$ with $\{u, v\} \subseteq Y$.

As a corollary of Lemma 2.3, we can observe that the following property holds.

Corollary 5.1 *Let $r : 2^V \to Z^+$ be a monotone set-function on V and X, Y be proper subsets of V with $p(X), p(Y) > 0$.*
(i) If (a) $X, Y \in \mathcal{A}^*$, (b) $X, Y \in \mathcal{B}^*$, or (c) $X \in \mathcal{A}^*$, $Y \in \mathcal{B}^*$, and $V = X \cup Y$, then $p(X) + p(Y) \leq p(X - Y) + p(Y - X) - 2d_G(X \cap Y, V - (X \cup Y))$. In particular, in the cases of (a) or (b), if the equality holds, then $R(X - Y) = r(X - Y)$ and $R(Y - X) = r(Y - X)$.
(ii) In all other cases, $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. □

From the symmetry of p, we can observe that all neighbors of s in H cannot be included in one dangerous set.

Lemma 5.2 *Let $p : 2^V \to Z^+$ be a symmetric function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. If $Y \subset V$ is dangerous, then $d_H(s, V - Y) \geq d_H(s, Y) - 1 > 0$.*

Proof: Since Y is dangerous and $p(Y) = p(V - Y)$, we have $d_H(s, Y) \leq p(Y) + 1 = p(V - Y) + 1 \leq d_H(V - Y) + 1$. From the definition of dangerous sets, it follows that $d_H(s, Y) \geq 2$. □

The next two lemmas show properties for proper sets Y with $d_H(s, Y) - p(Y) \leq 1$ and $p(Y) > 0$ (note that Y is not necessarily dangerous). We will be often referred to the next Lemma 5.3 in the subsequent arguments, when we observe that a dangerous set of $\mathcal{A}^*$ induces a connected graph, or that a dangerous set which does not induce a connected graph belongs to $\mathcal{B}^*$.

Lemma 5.3 *Let $r : 2^V \to Z^+$ be a monotone function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. For every set $Y \subset V$ of $\mathcal{A}^*$ with $d_H(s, Y) - p(Y) \leq 1$, $R(Y) \geq 2$, and $p(Y) > 0$, any set $\emptyset \neq Y' \subset Y$ satisfies $d_G(Y', Y - Y') \geq R(Y) - \lfloor \frac{d_H(Y)}{2} \rfloor$ (≥ 1).* □

The next lemma is often used under a situation where two crossing dangerous cuts Y_1, Y_2 satisfies $d_H(s, Y_1 \cap Y_2) > 0$. We call a set $Y \subset V$ with $d_H(s, Y) = p(Y) > 0$ *tight* (note that each tight set Y with $d_H(s, Y) \geq 2$ is dangerous).

Lemma 5.4 *Let $r : 2^V \to Z^+$ be a monotone function and $H = (V \cup \{s\}, E \cup F)$ be a p-extension of $G = (V, E)$. Let Y_1 and Y_2 be two sets with $d_H(s, Y_i) - p(Y_i) \leq 1$ and $p(Y_i) > 0$ for $i = 1, 2$, and $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) > 0$ such that Y_1 and Y_2 satisfy (i) $Y_1, Y_2 \in \mathcal{A}^*$ or (ii) $Y_1, Y_2 \in \mathcal{B}^*$. If Y_1 and Y_2 cross each other in H, then the following (a) – (d) hold:*
(a) $Y_1 - Y_2, Y_2 - Y_1 \in \mathcal{A}^*$.
(b) $d_H(s, Y_i) = p(Y_i) + 1$ for $i = 1, 2$.
(c) $d_H(s, Y_j - Y_k) = p(Y_j - Y_k)$ for $\{j, k\} = \{1, 2\}$. In particular, if $d_H(s, Y_j - Y_k) > 0$, $Y_j - Y_k$ is tight.
(d) $d_H(Y_1 \cap Y_2, (V \cup \{s\}) - (Y_1 \cup Y_2)) = 1$. □

5.1 Proof of Theorem 4.4

We first define a new operation called *hooking up*, which is a reverse operation of edge-splittings. We say that H' is obtained from H by *hooking up* an edge $(u,v) \in E(H-s)$ at s, if we construct H' by replacing an edge (u,v) with two edges (s,u) and (s,v) in H.

For proving Theorem 4.4, it suffices to show the following Theorem 5.5 and Lemma 5.6.

Theorem 5.5 *Let $r: 2^V \to Z^+$ be a monotone function satisfying (4.2) and $H = (V \cup \{s\}, E \cup F)$ be a critical p-extension of $G = (V,E)$. Assume that there is no admissible pair in H. Then the following (i) or (ii) hold:*
(i) $d_H(s) = 3$. After adding one edge incident to s, there is a complete admissible splitting.
(ii) $d_H(s) = 4$ and G has exactly two components C_1 and C_2 such that (a) $d_H(s,C_1) = 3$ and $d_H(s,C_2) = 1$, (b) every set $\emptyset \neq X \subseteq C_1$ satisfies $d_H(X) \geq 2$, and (c) every set $\emptyset \neq X \subseteq C_1$ with $d_H(X) = 2$ is a proper set of $\mathcal{A}$. □

Lemma 5.6 *Let H and r satisfy the assumption of Theorem 5.5 and $d_H(s) = 4$, and C_1 and C_2 be components in Theorem 5.5. Then for every edge $e = (u,v)$ in $G[V - C_1]$ (if exists), the graph H' obtained from H by hooking up the edge e has an admissible pair $\{e_1, e_2\}$ with $e_1 \in E_{H'}(s,C_1) = E_H(s,C_1)$ and $e_2 \in E_{H'}(s, V - C_1)$.* □

Before giving proofs of these theorem and lemma, we give a proof of Theorem 4.4 as its consequences.

PROOF of Theorem 4.4: (i) Let H_1 denote the graph from H by repeating admissible splittings as possible, E_1 denotes the set of split edges, and $G_1 = (V, E \cup E_1)$; the p_1-extension H_1 of G_1 has no admissible pair at s, where $p_1(X) = \max\{0, R(X) - d_{G_1}(X)\}$ for every $\emptyset \neq X \subset V$ and $p_1(\emptyset) = p_1(V) = 0$.

Theorem 5.5 implies that $d_{H_1}(s) \in \{0,3,4\}$. If $d_{H_1}(s) = 3$, then we can add one edge between s and V so that the resulting graph has a complete admissible splitting at s, by Theorem 5.5(i). If $d_{H_1}(s) = 4$, then after adding one edge connecting two components C_1 and C_2 satisfying (a) and (b) in Theorem 5.5(ii), we can obtain a complete admissible splitting at s (note that in the graph H' resulting from adding the edge, all neighbors of s is contained in one component in $H' - s$, and hence Theorem 5.5 ensures the existence of a complete admissible splitting in H'). Thus, in any case, after adding at most one edge in G or making the odd degree of s even, there is a complete admissible splitting at s.

(ii) Assume that $d_H(s)$ is even, because the case of odd $d_H(s)$ has been already seen in the above case of $d_{H_1}(s) = 3$. Since at least one of (P2*)–(P4*) does not hold, there are the following four possible cases:

(I) Every component C of G satisfies $d_H(s,C) \neq 1$.
(II) There are at least two components C of G with $d_H(s,C) = 1$.
(III) There is exactly one component C of G with $d_H(s,C) = 1$ where $\{(s,u)\} = E_H(s,C)$ holds. In H, $\{(s,u),(s,v)\}$ is admissible for some $(s,v) \in E_H(s, V - V_1) - \{(s,u)\}$.
(IV) There is exactly one component C of G with $d_H(s,C) = 1$ where $\{(s,u)\} = E_H(s,C)$ holds. There is no set $X \subseteq C$ with $u \in X$ and $d_H(s,X) = p(X)$.

Claim 5.7 *In the case (IV), there is a p-extension $H' = (V \cup \{s\}, E \cup (F - \{(s,u)\}) \cup \{(s,x)\})$ of G such that x is a vertex in some component $C' \neq C$ of G with $d_H(s,C') > 0$; H' belongs to the case (I).*

PROOF: Omitted due to space limitation. □

In the case (IV), according to this claim, replace H with H' which belongs to the case (I), and redenote H' by H. In the case (III), split (s,u) and (s,v) in H and denote the resulting graph by H'. Assume by contradiction that H has no complete splitting at s. Repeat admissible splittings as possible in H in the cases (I) and (II) and in H' in the case (III), and again consider H_1 defined as the above (i). Note that since $d_H(s)$ is even, $d_{H_1}(s) = 4$.

Then we have only to consider the cases where

$$G_1[V - C_1] \text{ contains no split edge in } E_1. \tag{5.1}$$

Consider the cases where $G_1[V - C_1]$ has a split edge $e \in E_1$. The graph H_2 obtained from H_1 by hooking up e has an admissible pair $\{e_1, e_2\}$ with $e_1 \in E_{H_2}(s,C_1)$ and $e_2 \in E_{H_2}(s, V - C_1)$ by Lemma 5.6. From the assumption, the graph H_3 obtained from H_2 by splitting e_1 and e_2 has no complete splitting, and has two components C'_1 and C'_2 satisfying (a) and (b) in Theorem 5.5. By $C_1 \subset C'_1$, we can see that the number of split edges in $H_3[V - C'_1]$ is less than that in $H_1[V - C_1]$. By repeating this observation, we can assume that $G_1[V - C_1]$ contains no split edge in E_1.

In the case (I), $d_H(s,C_2) = 1$ implies that $G[C_2]$ contains a split edge in E_1 and hence such H_1 satisfying (5.1) does not exist; in this case, H has a complete admissible splitting.

Consider the case (II). Let C', C'' denote components of G with $d_H(s,C') = d_H(s,C'') = 1$. By (5.1), $C' = C_2$ and $C'' \subseteq C_1$ without loss of generality. Then $d_H(C'') = 1 < 2$ contradicts Theorem 5.5(ii)(b). Hence also in the case (II), such H_1 does not exist.

Consider the case (III). Let C' denote the component containing v in H. If $d_{H'}(s, C \cup C') \neq 1$ in the graph H' obtained from H by splitting (s,u) and (s,v), then H' has no component C'' of $H' - s$ with $d_{H'}(s,C'') = 1$ and belongs to the case (I), which indicates that H' has a complete admissible splitting at s. Consider the case of $d_{H'}(s, C \cup C') = 1$; $d_H(s,C') = 2$. From the choice of (s,v), C' is not proper, since if C' is proper, then $C' \in \mathcal{C}_1$ would hold. By (5.1), in H_1, we have $C' \subseteq C_1$ and $d_{H_1}(C') = 2$, contradicting Theorem 5.5(ii)(c). Hence also in this case, such H_1 does not exist.

Consequently, in any case of (I)–(IV) such H_1 does not exist; H has a complete admissible splitting. □

In the rest of this section, we give a proof of Theorem 5.5. The proof of Lemma 5.6 is omitted due to space limitation. In (Nutov 2005, Proposition 5.3), it was shown that a critical extension of G which has no admissible pair has the following property if p is a symmetric skew-supermodular.

Theorem 5.8 (Nutov 2005) *Let $p: 2^V \to Z^+$ be a symmetric skew-supermodular set-function on V, and H be a critical p-extension. If there is no admissible pair in H, then p is $\{0,1\}$-valued.* □

For a graph $G = (V,E)$, every three sets X, Y, and Z satisfy the following inequality.

$$\begin{aligned} &d_G(X) + d_G(Y) + d_G(Z) \\ &\quad \geq d_G(X - Y - Z) + d_G(Y - X - Z) \\ &\qquad + d_G(Z - X - Y) + d_G(X \cap Y \cap Z) \\ &\qquad + 2d_G(X \cap Y \cap Z, V - (X \cup Y \cup Z)). \end{aligned} \tag{5.2}$$

PROOF of Theorem 5.5: Lemma 2.2 and Theorem 5.8 imply that p is $\{0,1\}$-valued, and hence the following claim holds (note that H is critical).

Claim 5.9 *(i) Every set* $X \subseteq V$ *satisfies* $d_G(X) \geq R(X) - 1$*. In particular, if* X *is dangerous, then* $d_G(X) = R(X) - 1$ *and* $d_H(s, X) = 2$*.*
(ii) $d_H(s, u) \leq 1$ *holds for every* $u \in V$*.* □

Observe that $d_H(s) \geq 3$ since $d_H(s) = 1$ would contradict the criticality of H and $d_H(s) = 2$ would contradict that no pair is admissible. There are the following two possible cases: (Case-1) $d_H(s) = 3$ and (Case-2) $d_H(s) \geq 4$.
(Case-1) Let u_0, u_1, u_2 be three distinct neighbours of s in H (these vertices exist by Claim 5.9 (ii)). Let H_1 be the graph obtained from H by adding one edge connecting s and u_0; $d_{H_1}(s, u_0) = 2$. Then we claim that $\{(s, u_0), (s, u_1)\}$ is admissible in H_1. Indeed, for any set Y containing u_0 and u_1 which is dangerous in H, we have $d_{H_1}(s, Y) = d_H(s, Y) + 1 = p(Y) + 2$, since Claim 5.9(i) implies that $d_G(Y) = R(Y) - 1$ and $d_H(s, Y) = p(Y) + 1$. Therefore, H_1 has a complete admissible splitting at s; the statement (i) is proved.
(Case-2) Let $u_0, u_1, u_2, u_3 \in V$ be four distinct neighbours of s in H. Let Y_i denote a dangerous set with $\{u_0, u_i\} \subseteq Y_i$, $i = 1, 2, 3$. Note that $E_H(s, Y_i) = \{(s, u_0), (s, u_i)\}$ by Claim 5.9, and hence we have $u_1 \in Y_1 - Y_2 - Y_3$, $u_2 \in Y_2 - Y_3 - Y_1$, and $u_3 \in Y_3 - Y_1 - Y_2$.

Claim 5.10 *(i) Each* $Y_i \in \mathcal{B}^*$ *holds and we have* $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$ *and* $d_H(s, Y_1 \cap Y_2 \cap Y_3) = 1$*, or (ii)* $\{Y_1, Y_2\} \subseteq \mathcal{A}^*$ *and* $Y_3 \in \mathcal{B}^*$ *without loss of generality,* $d_G(Y_3 - Y_1 - Y_2) = 0$*, and* $R(Y_1) = R(Y_2) = R(Y_3)$*.*

Proof: Without loss of generality, there are the following four possible cases:
(I) $Y_1, Y_2, Y_3 \in \mathcal{A}^*$.
(II) $Y_1, Y_2, Y_3 \in \mathcal{B}^*$.
(III) $Y_1 \in \mathcal{A}^*$, $Y_2, Y_3 \in \mathcal{B}^*$.
(IV) $Y_1, Y_2 \in \mathcal{A}^*$, $Y_3 \in \mathcal{B}^*$.

(I) Lemma 2.3(i) implies that $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_1 - Y_2 - Y_3) - 1 \geq R(Y_1) - 1$, $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_2 - Y_3 - Y_1) - 1 \geq R(Y_2) - 1$, $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_3 - Y_1 - Y_2) - 1 \geq R(Y_3) - 1$, and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1 \cap Y_2 \cap Y_3) - 1 \geq R(Y_1) - 1$. By (5.2) and Claim 5.9(i), it follows that $R(Y_1) - 1 + R(Y_2) - 1 + R(Y_3) - 1 = d_G(Y_1) + d_G(Y_2) + d_G(Y_3) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq 2R(Y_1) + R(Y_2) + R(Y_3) - 4$. Hence $R(Y_1) \leq 1$, contradicting (4.2). The case (I) does not occur.

(II) By $Y_1 \in \mathcal{B}^*$, $V - Y_1 \in \mathcal{A}^*$ holds and Lemma 2.3(i) implies that $Y_2 - Y_3 - Y_1 \in \mathcal{A}$ and $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_2 - Y_3 - Y_1) - 1 \geq R(V - Y_1) - 1 = R(Y_1) - 1$. Similarly, $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_2) - 1$ and $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_3) - 1$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) = \sum_{i=1}^{3} d_G(Y_i) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1) + R(Y_2) + R(Y_3) - 3$. Thus, every inequality turns out to be an equality, and hence $d_G(Y_1 \cap Y_2 \cap Y_3) = 0$. By $d_H(s, Y_1) = 2$ and $u_0 \in Y_1 \cap Y_2 \cap Y_3$, $d_H(s, Y_1 \cap Y_2 \cap Y_3) = 1$.

(III) Similarly to the above case, we have $d_G(Y_1 - Y_2 - Y_3) \geq R(Y_1) - 1$ and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq R(Y_1) - 1$ by $Y_1 \in \mathcal{A}^*$ and $d_G(Y_3 - Y_1 - Y_2) \geq R(Y_2) - 1$ and $d_G(Y_2 - Y_3 - Y_1) \geq R(Y_3) - 1$ by $Y_2, Y_3 \in \mathcal{B}^*$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) = \sum_{i=1}^{3} d_G(Y_i) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq 2R(Y_1) + R(Y_2) + R(Y_3) - 4$. Hence $R(Y_1) \leq 1$, contradicting (4.2). Thus, the case (III) does not occur.

(IV) Similarly to the above cases, we can observe that $d_G(Y_1 - Y_2 - Y_3) \geq \max\{R(Y_1), R(Y_3)\} - 1$, $d_G(Y_2 - Y_3 - Y_1) \geq \max\{R(Y_2), R(Y_3)\} - 1$, and $d_G(Y_1 \cap Y_2 \cap Y_3) \geq \max\{R(Y_1), R(Y_2)\} - 1$. Again by (5.2), it follows that $\sum_{i=1}^{3}(R(Y_i) - 1) \geq d_G(Y_1 - Y_2 - Y_3) + d_G(Y_2 - Y_3 - Y_1) + d_G(Y_3 - Y_1 - Y_2) + d_G(Y_1 \cap Y_2 \cap Y_3) \geq \max\{R(Y_1), R(Y_2)\} + \max\{R(Y_2), R(Y_3)\} + \max\{R(Y_3), R(Y_1)\} - 3$. Thus, every inequality turns out to be an equality, and hence $d_G(Y_3 - Y_1 - Y_2) = 0$ and $R(Y_1) = R(Y_2) = R(Y_3)$. □

Claim 5.11 *There is at least one dangerous set of* $\mathcal{A}^*$ *in* H*.*

Proof: Assume by contradiction that every dangerous set in H belongs to $\mathcal{B}^*$. By Claim 5.10, for every $(s, u) \in E_H(s, V)$, there is a component C_u of G with $E_H(s, C_u) = \{(s, u)\}$. Let Y be a dangerous set and u, v be two neighbors of s with $u, v \in V - Y$ (such u, v exist because $d_H(s) \geq 4$ and $d_H(s, Y) = 2$ by Claim 5.9(i)). Then $Y \cap C_u \neq \emptyset$ holds, since otherwise $Y \in \mathcal{B}^*$ implies that $C_u \in \mathcal{A}$ and $1 = d_H(s, C_u) \geq p(C_u) = R(C_u)$, contradicting (4.2). Hence $d_G(Y \cup C_u) = d_G(Y) - d_G(Y, C_u - Y) \leq d_G(Y) - 1 = R(Y) - 2$ by $d_G(C_u) = 0$ and Claim 5.9(i). On the other hand, by $v \notin Y \cup C_u$ and $Y \in \mathcal{B}^*$, Lemma 2.3 implies that $R(Y \cup C_u) \geq R(Y)$. It follows that $d_G(Y \cup C_u) \leq R(Y \cup C_u) - 2$, contradicting Claim 5.9(i). □

Rechoose u_i and Y_i so that $Y_1 \in \mathcal{A}^*$. Then $d_H(s) = 4$ holds. Indeed, if $d_H(s) \geq 5$, then some three dangerous sets containing u_0 satisfy the cases (I) or (III) in the proof of Claim 5.10, in both cases of $Y_4 \in \mathcal{A}^*$ and $Y_4 \in \mathcal{B}^*$, where Y_4 denotes a dangerous set containing u_0 and u_4 with some neighbor $u_4 \notin \{u_0, u_1, u_2, u_3\}$ of s. According to Claim 5.10, let $Y_2 \in \mathcal{A}^*$ and $Y_3 \in \mathcal{B}^*$ without loss of generality. Let $Y_{ij} = V - Y_k$ with $\{i, j, k\} = \{1, 2, 3\}$ and $i < j$. Then Y_{ij} is also dangerous because Y_{ij} is clearly proper and satisfies $d_H(s, Y_{ij}) = 4 - d_H(s, Y_k) = 2$ and $d_G(Y_{ij}) = d_G(Y_k) = R(Y_k) - 1 = R(Y_{ij}) - 1$. Hence, Y_{12} is a dangerous set of $\mathcal{A}^*$ and Y_{23} and Y_{13} are dangerous sets of $\mathcal{B}^*$.

Lemma 5.3 implies that $Y_1 \cup Y_2 \cup Y_{12} = V - (Y_3 - Y_1 - Y_2)$ induces a connected graph. Claim 5.10 implies that $d_G(Y_3 - Y_1 - Y_2) = 0$. It follows that $Y_1 \cup Y_2 \cup Y_{12}$ is a component of G containing $\{u_0, u_1, u_2\}$, and that $Y_1 \cup Y_2 \cup Y_{12}$ and the component of G containing u_3 correspond to C_1 and C_2 of the statement of this theorem, respectively.

We next show the statement (ii)(b); every set $X \subseteq C_1$ satisfies $d_H(X) \geq 2$. Let $C_1 = Y_1 \cup Y_2 \cup Y_{12}$. We first claim that $C_1 = Y_1 \cup Y_2$.

Claim 5.12 $C_1 = Y_1 \cup Y_2$.

Proof: Omitted due to space limitation. □

For proving (ii)(b), assume by contradiction that there is a set $X \subseteq C_1$ with $d_H(X) = 1$. Clearly, $d_H(s, X) = 0$ and $d_G(X) = 1$ since $d_H(s, C_1) = 3$ and C_1 induces a connected graph. Moreover, X is not proper since otherwise $0 = d_H(s, X) \geq p(X) \geq R(X) - d_G(X) = R(X) - 1$, contradicting (4.2). Hence, X is not contained in any of Y_1 and Y_2; $X \cap (Y_1 - Y_2) \neq \emptyset \neq X \cap (Y_2 - Y_1)$ by Claim 5.12. Now by applying Lemma 5.4 to Y_1 and Y_2, both of $Y_1 - Y_2$ and $Y_2 - Y_1$ are tight sets of $\mathcal{A}^*$ (note that $d_H(s, Y_1 - Y_2) = p(Y_1 - Y_2) > 0, d_H(s, Y_2 - Y_1) = p(Y_2 - Y_1) > 0$). Lemma 5.3 implies that $G[Y_1 - Y_2]$ and $G[Y_2 - Y_1]$ are both connected. Then it is not difficult to see that $d_G(X) = 1$ would contradict the connectedness of $G[Y_1 - Y_2]$ or $G[Y_2 - Y_1]$.

We finally show (ii)(c); every set $\emptyset \neq X \subseteq C_1$ with $d_H(X) = 2$ belongs to $\mathcal{A}$. Assume by contradiction that $X \subseteq C_1$ does not belong to $\mathcal{A}$. By Lemma 2.3(i), X cannot be included in any of Y_1 and Y_2. Hence, we can assume that $X \cap (Y_1 - Y_2) \neq \emptyset \neq X \cap (Y_2 - Y_1)$.

By $d_H(X) = 2$ and $d_H(C_1) \geq 3$, we have $C_1 - X \neq \emptyset$. Since $G[C_1]$ is connected, it follows that $d_G(X) \geq 1$, from which $d_H(s, X) \leq 1$. This implies that X and Y_1 cross each other in H. From (2.2) and $X - Y_1 \subseteq Y_2$, we have $(R(Y_1) - 1 + 2) + 2 = d_H(Y_1) + d_H(X) = d_H(Y_1 - X) + d_H(X - Y_1) + 2d_H(X \cap Y_1, (V \cup s) - X - Y_1) \geq R(Y_1 - X) + R(X - Y_1) \geq R(Y_1) + R(Y_2)$ (note that $d_H(X') = d_H(s, X') + d_G(X') \geq R(X')$ holds for every $X' \subseteq V$ by $d_H(s, X') \geq p(X')$). Now observe that $R(Y_2) \geq 2$ by (4.2) and that $d_H(X - Y_1) \geq R(Y_2) = R(Y_1)$ by Claim 5.10. It follows that $d_H(X \cap Y_1, V \cup \{s\} - X - Y_1) = 0$ and $d_H(Y_1 - X) \leq 3$. Hence we have $Y_1 - Y_2 - X \neq \emptyset \neq (Y_1 \cap Y_2) - X$ by $d_H(s, Y_1 - Y_2) > 0$ and $d_H(s, Y_1 \cap Y_2) > 0$. By these and $X \cap (Y_1 - Y_2) \neq \emptyset$, $Y_1 - X$ and $Y_1 - Y_2$ cross each other in H. From (2.2) and $d_H(Y_1 - X) \leq 3$, it follows that $d_H(Y_1 - Y_2) + 3 \geq d_H(Y_1 - Y_2) + d_H(Y_1 - X) \geq d_H((Y_1 - Y_2) \cap X) + d_H(Y_1 \cap Y_2 - X) + 2d_H(s, Y_1 - Y_2 - X) \geq R((Y_1 - Y_2) \cap X) + R(Y_1 \cap Y_2 - X) + 2 \geq R(Y_1 - Y_2) + R(Y_2) + 2$ (note that $R((Y_1 - Y_2) \cap X) \geq R(Y_1 - Y_2)$ and $R(Y_1 \cap Y_2 - X) \geq R(Y_2)$ by $Y_1 - Y_2, Y_2 \in \mathcal{A}^*$). It follows from $R(Y_2) \geq 2$ that $d_H(Y_1 - Y_2) \geq R(Y_1 - Y_2) + 1$. Now as observed in the above, $d_H(s, Y_1 - Y_2) = p(Y_1 - Y_2) > 0$ and hence $d_H(Y_1 - Y_2) = R(Y_1 - Y_2)$, a contradiction. □

5.2 Proof of the sufficiency Theorem 4.3

Let $r : 2^V \to Z^+$ be a monotone function on V and H be a p-extension of $G = (V, E)$ with property (P*). In this subsection, we prove that G has property (P). By (P4*), for each $(s, v) \in E_H(s, V_2 - C^*)$ there is a dangerous set Y with $\{u^*, v\} \subseteq Y$, which will play a role as a cut Y_X in Definition 3.2 in the subsequent arguments. Note that any proper set X with $X \cap C^* = \emptyset$ belongs to $\mathcal{A}^*$, since if $X \in \mathcal{B}$, then $C^* \in \mathcal{A}$ and $1 = d_H(s, C^*) \geq p(C^*) = R(C^*) \geq 2$ by (4.2), a contradiction. Hence, each $C \in \mathcal{C}_1$ satisfies $C \in \mathcal{A}^*$. We first show properties of such dangerous sets in Lemma 5.13, and show by Lemma 5.14 that G has property (P).

Lemma 5.13 *Let H be a p-extension of $G = (V, E)$ with property (P*), and $(s, v) \in E_H(s, V_2 - C^*)$ and Y_v be a dangerous set with $\{u^*, v\} \subseteq Y_v$ (such Y_v exists by the property (P4*)). Then*
(i) $d_H(s, V_2 - Y_v) \geq 1$ holds.
(ii) For some $(s, w) \in E_H(s, V_2 - C^*) - \{(s, v)\}$, Y_v and Y_w cross each other in H, where Y_w denotes a dangerous cut with $\{u^*, w\} \subseteq Y_w$ in H. Moreover, $v \in Y_v - Y_w$ and $Y_v \subset V - V_1$ hold and $Y_v - Y_w$ is a tight set of $\mathcal{A}^*$ with $Y_v - Y_w \subseteq V_2$.
(iii) $Y_v \cup C^*$ is a dangerous set of $\mathcal{B}^*$.

PROOF: Note that $Y_v \in \mathcal{B}^*$ holds by Lemma 5.3 since Y_v does not induce a connected graph. Also note that $d_H(s, V_2) \geq 4$ holds since $d_H(s, V_2)$ is even by the property (P1*) and the property that $d_H(s, V_1)$ is even, and $d_H(s, V_2 - C^*) \neq 1$ holds by the property (P2*).

(i) Assume by contradiction that $d_H(s, V_2 - Y_v) = 0$ holds. Let Y'_v be a dangerous set with $Y_v \subseteq Y'_v$ such that no $Y'' \supset Y'_v$ is dangerous. Note that $Y'_v \in \mathcal{B}^*$ and $d_H(s, V_2 - Y'_v) = 0$ also hold. We have $p(Y'_v) \geq d_H(s, Y'_v) - 1 \geq d_H(s, V_2) - 1 \geq 3$ holds, from which $R(Y'_v) \geq 3$. Lemma 5.2 and $d_H(s, Y'_v) \geq 4$ imply that $d_H(s, V - Y'_v) \geq 3$. It follows that there exist at least two sets $C_1, C_2 \in \mathcal{C}_1$ with $d_H(s, C_i - Y'_v) > 0$ for $i = 1, 2$. We have $C_1 \cap Y'_v \neq \emptyset$, since otherwise $C_1 \subseteq V - Y'_v \in \mathcal{A}^*$ and Lemma 2.3(i) imply that $2 = R(C_1) \geq R(V - Y'_v) = R(Y'_v) \geq 3$, a contradiction. Now by $C_1 \in \mathcal{A}^*$, every $\emptyset \neq X \subseteq C_1$ satisfies $d_H(X) = d_H(s, X) + d_G(X) \geq R(X) \geq R(C_1) \geq 2$. This indicates that $d_H(Y'_v) = d_H(Y'_v \cap C_1) + d_H(Y'_v - C_1) \geq 2 + d_H(Y'_v - C_1) = d_H(Y'_v \cup C_1)$. It follows from Lemma 2.3(ii) and $d_H(s, C_2 - Y'_v) > 0$ that $Y'_v \cup C_1 \in \mathcal{B}$ and $R(Y'_v) \leq R(Y'_v \cup C_1)$. This indicates that $Y'_v \cup C_1$ is also dangerous by $d_H(Y'_v \cup C_1) \leq d_H(Y'_v) \leq R(Y'_v) + 1 \leq R(Y'_v \cup C_1) + 1$. This contradicts the maximality of Y'_v.

(ii) Let Y'_v be a dangerous set with $\{u^*, v\} \subseteq Y'_v$ and $Y_v \subseteq Y'_v$ such that no $Y'' \supset Y'_v$ is dangerous in H. By (i), $d_H(s, V_2 - Y'_v) > 0$ holds. Let $w \in V_2 - Y'_v$ be a vertex with $d_H(s, w) > 0$ and Y_w be a dangerous set with $\{u^*, w\} \subseteq Y_w$. Then Y'_v and Y_w cross each other in H since we have $u^* \in Y'_v \cap Y_w$, $w \in Y_w - Y'_v$, and $Y'_v - Y_w \neq \emptyset$ by the maximality of Y'_v. Note that $Y_w \in \mathcal{B}^*$. Lemma 5.4 implies that $d_H(s, Y'_v \cap Y_w) = 1$, and it follows from $u^* \in Y'_v \cap Y_w$ that $v \in Y_v - Y_w$. Hence, Y_v and Y_w also cross each other in H.

Again by Lemma 5.4, we have $p(Y_v - Y_w) = d_H(s, Y_v - Y_w) > 0$, and hence $Y_v - Y_w$ is a tight set of $\mathcal{A}^*$ and Lemma 5.3 implies that $G[Y_v - Y_w]$ is connected; $Y_v - Y_w \subseteq V_2$. Similarly, $G[Y_w - Y_v]$ is connected. Finally, we prove that $Y_v \cap Y_w \cap V_1 = \emptyset$ in order to show that $Y_v \subset V - V_1$ (note that $V - V_1 - Y_v \neq \emptyset$ holds by $d_H(s, V_2 - Y_v) > 0$). Assume by contradiction that $Y_v \cap Y_w \cap C \neq \emptyset$ holds for some $C \in \mathcal{C}_1$. From $d_H(s, V_2 - Y_v) > 0$, $d_H(s, V_2 - Y_w) > 0$, and the similar arguments in the above (i), it is not difficult to see that $Y_v \cup C$ and $Y_w \cup C$ are both dangerous sets of $\mathcal{B}^*$ and cross each other in H. Then $d_H(s, (Y_v \cap Y_w) \cup V_1) \geq 3$ would contradict Lemma 5.4.

(iii) Omitted due to space limitation. □

Lemma 5.14 *If $H = (V \cup \{s\}, E \cup F)$ is a p-extension of $G = (V, E)$ with property (P*), then G has property (P).*

PROOF: Lemma 5.13 implies that for each $v \in V[F] - V_1 - \{s, u^*\}$, there are two proper sets $X_v \subset V - V_1$ and $Y_v \subset V - V_1$ with $v \in X_v \subseteq Y_v$ satisfying the following (a) and (b).

(a) X_v is a tight set of $\mathcal{A}^*$, and no set $\emptyset \neq X' \subset X_v$ with $v \in X'$ satisfies this property.

(b) Y_v satisfies $u^* \in Y_v$ and $C^* \subseteq Y_v \subset V - V_1$ (by (ii)(iii) in Lemma 5.13) and is a dangerous set of $\mathcal{B}^*$. Let X_{u^*} be a tight set with $u^* \in X_{u^*} \subseteq C^*$ such that no set $X' \subset X_{u^*}$ satisfies this property (such X_{u^*} exists from the property (P3*)). Let $\mathcal{X}$ be the family of all sets X_v, $v \in V[F] - \{s\} - V_1$ such that $\cup_{X \in \mathcal{X}} X \supseteq V[F] - \{s\} - V_1$ and $X_v \in \mathcal{X}$ does not satisfy $X_v \subset X$ for any $X \in \mathcal{X}$, and $\mathcal{Y}$ be the family of the corresponding Y_v. We will show that $\alpha(G, r)$ is even, implying (P1), and the family $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V satisfying $\sum_{X \in \mathcal{X} \cup \mathcal{C}_1} p(X) = \alpha(G, r)$ and (P2) and (P3), which proves the lemma.

We claim that

$$\mathcal{X} \text{ is a subpartition of } V - V_1. \tag{5.3}$$

Assume by contradiction that there are two sets $X_u, X_v \in \mathcal{X}$ which cross each other in H. By $X_u, X_v \in \mathcal{A}^*$ and Corollary 5.1, we have $0 \geq d_H(s, X_u) - p(X_u) + d_H(s, X_v) - p(X_v) \geq d_H(s, X_u - X_v) - p(X_u - X_v) + d_H(s, X_v - X_u) - p(X_v - X_u) + 2d_G(X_u \cap X_v, V - X_u - X_v) + 2d_H(s, X_u \cap X_v) \geq 0$. It follows that $d_H(s, X_u - X_v) = p(X_u - X_v)$, $d_H(s, X_v - X_u) = p(X_v - X_u)$, and $d_H(X_u \cap X_v, (V \cup \{s\}) - X_u - X_v) = 0$. Hence $u \in X_u - X_v$ and $p(X_u - X_v) = d_H(s, X_u - X_v) > 0$. Now $X_u - X_v \in \mathcal{A}$ holds by Lemma 2.3(i). Since $X_u - X_v \notin \mathcal{B}$, $X_u - X_v \in \mathcal{A}^*$. Thus, $X_u - X_v$ is also tight of $\mathcal{A}^*$, contradicting the minimality of X_u.

Now each $C \in \mathcal{C}_1$ is tight since $2 = d_H(s, C) \geq p(C) = R(C) \geq 2$ holds by (4.2). Hence, by (5.3), $\mathcal{X} \cup$

$\mathcal{C}_1$ is a subpartition of V and a family of tight sets such that $V[F] - \{s\} \subseteq \cup_{X \in \mathcal{X} \cup \mathcal{C}_1} X$. Since $|F| = \alpha(G, r)$ holds by Theorem 4.1, $\sum_{X \in \mathcal{X} \cup \mathcal{C}_1} p(X) = d_H(s, V) = |F| = \alpha(G, r)$. Since $|F|$ is even, $\alpha(G, r)$ is even. Moreover, $\mathcal{X} \cup \mathcal{C}_1$ is a subpartition of V satisfying (P2) by taking $X^* = X_{u^*}$. Now for every dangerous set $Y \in \mathcal{Y}$ which does not cross with any $X \in \mathcal{X}$ in H, we have $\sum_{X' \in \mathcal{X}, X' \subseteq Y} p(X') = \sum_{X' \in \mathcal{X}, X' \subseteq Y} d_H(s, X') = d_H(s, Y) \leq p(Y) + 1$. Moreover, note that each $Y \in \mathcal{Y}$ satisfies $V - V_1 - Y \neq \emptyset$ by $V_2 - Y \neq \emptyset$. Therefore, by regarding $\mathcal{C}_1$ as $\mathcal{X}_1$ in Definition 3.2, in order to show that $\mathcal{X} \cup \mathcal{C}_1$ satisfies (P3), it suffices to prove that for any $X_u \in \mathcal{X}$ with $u \neq u^*$, there is a set $Y_w \in \mathcal{Y}$ with $X_u \subseteq Y_w$ such that for any set $X \in \mathcal{X}$, Y_w and X do not cross each other in H (note that each $Y \in \mathcal{Y}$ satisfies $C \cap Y = \emptyset$ for any $C \in \mathcal{C}_1$ by $Y \subset V - V_1$). For this, we show that

if there is a set $Y_u \in \mathcal{Y}$ which crosses with some $X_v \in \mathcal{X}$ in H, $v \neq u^*$ and $Y_u \subseteq Y_v$. (5.4)

Since each $Y \in \mathcal{Y}$ satisfies $X_{u^*} \subseteq C^* \subseteq Y$, $v \neq u^*$ holds. Assume by contradiction that $Y_u - Y_v \neq \emptyset$. By $X_v - Y_u \neq \emptyset \neq X_v \cap Y_u$, Y_u and Y_v cross each other in H. From Lemma 5.4, it follows that $Y_v - Y_u \in \mathcal{A}^*$, $d_H(s, Y_v - Y_u) = p(Y_v - Y_u)$, and $d_H(s, u^*) = d_H(Y_u \cap Y_v, V \cup \{s\} - Y_u - Y_v) = 1$. Hence we have $v \in X_v - Y_u$, from which $X_v \cap (Y_v - Y_u) \neq \emptyset$ holds and $Y_v - Y_u$ is tight. Note that $X_v - (Y_v - Y_u) \neq \emptyset$ holds since X_v and Y_u cross each other in H. Moreover, $(Y_v - Y_u) - X_v \neq \emptyset$ holds since if $Y_v - Y_u \subseteq X_v$ holds, then the tight set $Y_v - Y_u$ contradicts the minimality of X_v. This means that X_v and $Y_v - Y_u$ cross each other in H. Now $d_H(X_v \cap (Y_v - Y_u), V \cup \{s\} - X_v - (Y_v - Y_u)) > 0$ holds by $v \in X_v - Y_u$. By applying Lemma 5.4 to X_v and $Y_v - Y_u$, we have $d_H(s, X_v) = p(X_v) + 1$, contradicting that X_v is tight (note that X_v and $Y_v - Y_u$ are both tight sets of $\mathcal{A}^*$). Hence (5.4) holds. □

5.3 Step 2

According to the proof of Theorems 4.4, Step 2 of algorithm M-AUG is described as follows.

Step 2: (1) Check whether H has property (P*).
(2) The case where H has property (P*): Repeat admissible splittings as possible. In the resulting graph, after adding one edge between C_1 and C_2 according to the case of $d_{H_1}(s) = 4$ in the proof of Theorem 4.4(i), find a complete admissible splitting (note that $d_H(s)$ is even from the property (P1*)). Halt after outputting the set E^* of all edges added to G as an optimal solution, where $|E^*| = \lceil \alpha(G, r)/2 \rceil + 1$.
(3) The case where H does not have property (P*):
(3-1) If $d_H(s)$ is odd, then according to the proof of Theorem 4.4(i), find a complete admissible splitting by adding one edge incident to s and halt after outputting the set E^* of all edges added to G as an optimal solution, where $|E^*| = \lceil \alpha(G, r)/2 \rceil$.
(3-2) Otherwise one of the cases (I)–(IV) in the proof of Theorem 4.4(ii) hold. In the case of (IV), we replace one edge incident to s so that the resulting graph belongs to the case (I), according to Claim 5.7. In the case of (III), first split the edges (s, u) and (s, v) in H.

After that, in all cases repeat admissible splittings as possible. If the resulting graph H_1 still has an edge incident to s, then according to the statements immediately after (5.1), find a complete admissible splitting while hooking up some edges in $H_1[V - C_1]$ and resplitting (note that the proof of Theorem 4.4(ii) implies that $H_1[V - C_1]$ has a split edge and that hooking up and resplitting operations can find a complete admissible splitting). Halt after outputting the set E^* of all edges added to G, where $|E^*| = \lceil \alpha(G, r)/2 \rceil$. □

Finally we show that algorithm M-AUG can be implemented to run in $O(n^4(m + n \log n + q))$ time. Note that H satisfies (4.1) if and only if $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\} \geq 0$. We can prove the following lemma by using the family of all *extreme sets* (Nagamochi 2003), where in G, a set $\emptyset \neq X \subset V$ is called *extreme* if any $\emptyset \neq X' \subset X$ satisfies $d_G(X') > d_G(X)$.

Lemma 5.15 *It can be checked in $O(n^2(m + n \log n + q))$ time whether a given H satisfies (4.1) or not. Moreover, if H violates (4.1), then $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\}$ can be obtained in the same time.* □

Proof: Let $\mathcal{Z}(H)$ denote the family of all extreme sets in H. It is known that $\mathcal{Z}(H)$ is laminar and hence $|\mathcal{Z}(H)| = O(n(H))$. It was shown in (Nagamochi 2003) that $\mathcal{Z}(H)$ can be found in $O(m(H)n(H) + n(H)^2 \log n(H))$ time. Note that $m(H) \leq m(G) + n(G)$ and $n(H) = n(G) + 1$.

Let $H(v)$ denote the graph obtained from H by adding $\max\{r(u) \mid u \in V\}$ multiple edges to $E_H(s, v)$ for a vertex $v \in V$, and $\mathcal{Z}^s(H(v))$ denote the family of extreme sets $X \in \mathcal{Z}(H(v))$ in $H(v)$ with $s \in X$. For a given H, let $g(H) = \min\{\min\{d_H(X) - R(X) \mid X \in \mathcal{Z}(H), s \notin X\}, \min\{d_H(X) - R(X - s) \mid X \in \mathcal{Z}^s(H(v)), v \in V\}\}$. Note that given $\mathcal{Z}(H)$ and $\mathcal{Z}^s(H(v))$, $v \in V$, we obtain $g(H)$ by computing $d_H(X) - R(X)$ or $d_H(X) - R(X - s)$ $O(n^2)$ times; $O(n^2)$ times computation of r suffices. For proving this lemma, we will show that H satisfies (4.1) if and only if $g(H) \geq 0$, and that if H violates (4.1), then $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\} = g(H) < 0$.

For this, we first show by Claims 5.16 and 5.17 that $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\} \geq g(H)$.

Claim 5.16 *Every proper set $X \subset V$ of $\mathcal{A}$ satisfies $d_H(X) - r(X) \geq d_H(X') - r(X')$ for some $X' \in \mathcal{Z}(H)$ with $X' \subseteq X$.*

Proof: From the definition of extreme sets, there is an extreme set $Y \in \mathcal{Z}(H)$ with $Y \subseteq X$ and $d_H(Y) \leq d_H(X)$. By the monotonicity of r, $r(Y) \geq r(X)$. Hence, $d_H(X) - r(X) \geq d_H(Y) - r(Y)$. □

Claim 5.17 *Assume that $\min\{d_H(X) - r(V - X) \mid X \in \mathcal{B}\} < 0$. Then, every proper set $X \subset V$ of $\mathcal{B}$ satisfies (a) $d_H(X) - r(V - X) \geq d_H(X') - r(X')$ for some $X' \in \mathcal{Z}(H)$ with $X' \subseteq V - X$ or (b) $d_H(X) - r(V - X) \geq d_H(X') - r(X' - s)$ for some $X' \in \mathcal{Z}^s(H(v))$ and $v \in V$.*

Proof: Let $X \subset V$ be a proper set of $\mathcal{B}$ such that $d_H(X) - r(V - X) = \min\{d_H(X') - r(V - X') \mid X' \in \mathcal{B}\}$ and any set $V \neq X'' \supset X$ satisfies $d_H(X'') - r(V - X'') > d_H(X) - r(V - X)$ (note that each $V \neq X'' \supset X$ belongs to $\mathcal{B}$). Let $\overline{X} = V - X$. Note that $\overline{X} \in \mathcal{A}$. By $d_H(X) = d_H(\overline{X} \cup \{s\})$, we have $d_H(X) - r(V - X) = d_H(\overline{X} \cup \{s\}) - r(\overline{X}) = \min\{d_H(X' \cup \{s\}) - r(X') \mid X' \in \mathcal{A}\}$, and any set $\emptyset \neq X'' \subset \overline{X}$ satisfies $d_H(X'' \cup \{s\}) - r(X'') > d_H(\overline{X} \cup \{s\}) - r(\overline{X})$.

First we consider the case where some $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X') \leq d_H(\overline{X} \cup \{s\})$. Since $\overline{X} \in \mathcal{A}$, we have $X' \in \mathcal{A}$ and hence $r(X') \geq r(\overline{X})$ by the monotonicity of r. It follows that $d_H(\overline{X} \cup \{s\}) -$

$r(\overline{X}) \geq d_H(X') - r(X')$. Claim 5.16 implies that $d_H(X') - r(X') \geq d_H(Y) - r(Y)$ for some $Y \in \mathcal{Z}(H)$ with $Y \subseteq X'$.

Next consider the case where some $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X' \cup \{s\}) \leq d_H(\overline{X} \cup \{s\})$. Similarly to the above, $r(X') \geq r(\overline{X})$. Hence, $d_H(\overline{X} \cup \{s\}) - r(\overline{X}) \geq d_H(X' \cup \{s\}) - r(X')$, contradicting the minimality of $\overline{X}$.

Finally, we consider the case where every $\emptyset \neq X' \subset \overline{X}$ satisfies $d_H(X') > d_H(\overline{X} \cup \{s\})$ and $d_H(X' \cup \{s\}) > d_H(\overline{X} \cup \{s\})$. Let $u \in \overline{X}$ (note that $\overline{X} \neq \emptyset$). In $H(u)$, we can observe that $\overline{X} \cup \{s\} \in \mathcal{Z}^s(H(u))$ or $d_H(X) = d_H(\overline{X} \cup \{s\}) \geq r(V - X)$. Indeed, if $d_{H(u)}(s) > d_H(\overline{X} \cup \{s\}) = d_{H(u)}(\overline{X} \cup \{s\})$, then every set $X' \subset \overline{X} \cup \{s\}$ satisfies $d_{H(u)}(X') > d_{H(u)}(\overline{X} \cup \{s\})$, and otherwise then $d_H(\overline{X} \cup \{s\}) \geq \max\{r(w) \mid w \in V\} \geq r(V - X)$ (note that from the monotonicity of r, $\max\{r(X) \mid X \subseteq V\} = \max\{r(v) \mid v \in V\}$). □

Clearly, if $g(H) \geq 0$, H satisfies (4.1). Consider the case of $g(H) < 0$. Since every $X \subseteq V$ with $X \notin \mathcal{A} \cup \mathcal{B}$ satisfies $R(X) = 0$ and $d_H(X) - R(X) \geq 0$, we have $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\} = g(H) < 0$. H violates (4.1) and $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\}$ is also obtained. □

It suffices to show that the following (A) (resp. (B)) can be done by computing $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\}$ at most n times (resp. once):

(A) The computation of a critical p-extension of a given G.

(B) The computation of how many pairs of $\{(s, u), (s, v)\}$ are admissible for a given pair $\{u, v\} \subseteq V$ of two vertices in a p-extension H of G.

Indeed, Step 2(1) can be done by the computation (B) for $O(n)$ pairs, a sequence of greedy admissible splittings in Step 2(2)(3) can be done by the computation (B) for $O(n^2)$ pairs, and the hooking up operations in Step 2(3-2) are executed at most n times (since the statements immediately after (5.1) indicates that one hooking up decreases $|V - C_1|$ at least by one).

(A) A critical p-extension of G can be obtained as follows. First we add $\max\{r(v) \mid v \in V\}$ edges between s and each $v \in V$. From the monotonicity of r, $\max\{r(X) \mid X \subseteq V\} = \max\{r(v) \mid v \in V\}$, and hence the resulting graph H' is a p-extension of G. After that, for each $v \in V$, after deleting all edges between s and v, we check whether the resulting graph H'' satisfies (4.1) or not. If not, we add $-\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_{H''}(X) - R(X)\}$ edges between s and v in H''. Thus, a critical p-extension of G can be found by computing $\min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_H(X) - R(X)\}$ for some H at most n times.

(B) Given a p-extension H of G, we can check how many pairs of $\{(s, u), (s, v)\}$ can be split as follows. This can be done by checking whether the resulting graph H' satisfies (4.1) or not after splitting $\min\{d_H(s, u), d_H(s, v)\}$ pairs $\{(s, u), (s, v)\}$. If (4.1) is violated, then we have only to hook up $\lceil -\frac{1}{2} \min_{\emptyset \neq X \subset V, X \in \mathcal{A} \cup \mathcal{B}} \{d_{H'}(X) - R(X)\} \rceil$ pairs in H'.

6 Concluding Remarks

In this paper, given a graph $G = (V, E)$ and a monotone function $r : 2^V \to Z^+$, we considered the problem of asking to augment G by adding a smallest number of new edges F such that the resulting graph $G+F$ satisfies $d_{G+F}(X) \geq r(X)$ for every $\emptyset \neq X \subset V$. We have shown that the problem can be solved in $O(n^4(m + n \log n + q))$ time under the assumption that $r(X) \geq 2$ holds for every $X \subseteq V$ whenever $r(X) > 0$. It is a future work to consider RECAP with a more general R, such as one including both of LECAP and NAECAP.

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