

Computing a Minimum-Dilation Spanning Tree is NP-hard

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Abstract

Given a set S of n points in the plane, a minimum-dilation spanning tree of S is a tree with vertex set S of smallest possible dilation. We show that given a set S of n points and a dilation $\delta > 1$, it is NP-hard to determine whether a spanning tree of S with dilation at most δ exists.

1 Introduction

A *geometric network* is a weighted undirected graph whose vertices are points in \mathbb{R}^d , and in which the weight of an edge is the Euclidean distance between its endpoints. Geometric networks have many applications: most naturally, many communication networks (road networks, railway networks, telephone networks) can be modelled as geometric networks.

In a geometric network $G = (S, E)$ on a set S of n points, the *graph distance* $d_G(u, v)$ of $u, v \in G$ is the length of a shortest path from u to v in G . Some applications require a geometric network for a given set S of points that includes a relatively short path between every two points in S . More precisely, we consider the factor by which the graph distance $d_G(u, v)$ differs from the Euclidean distance $|uv|$. This factor is called the *dilation* Δ of the pair (u, v) in G , and is formally expressed as:

$$\Delta_G(u, v) := \frac{d_G(u, v)}{|uv|}$$

The dilation or stretch factor $\Delta(G)$ of a graph is the maximum dilation over all vertex pairs:

$$\Delta(G) := \max_{\substack{u, v \in S \\ u \neq v}} \Delta_G(u, v) = \max_{\substack{u, v \in S \\ u \neq v}} \frac{d_G(u, v)}{|uv|}.$$

A network G is called a t -spanner if $\Delta(G) \leq t$.

An obvious 1-spanner is the complete graph. It has optimal dilation and is easy to compute, but for many applications its high cost is unacceptable. Therefore one usually seeks to construct networks that do not only have small dilation, but also have properties such as a low number of edges, a low total edge weight or a low maximum vertex degree. Such networks find applications in, for example, robotics, network topology design, broadcasting, design of parallel machines and distributed systems, and metric space searching.

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Therefore there has also been considerable interest from a theoretical perspective [4, 10].

In this paper we focus on spanners that have small dilation and few edges. Several algorithms have been published to compute a $(1 + \varepsilon)$ -spanner with $O(n)$ edges for any given set of points S [2, 8, 9, 11] and any $\varepsilon > 0$. Farshi and Gudmundsson did an experimental study of such algorithms [5].

Although the number of edges in the spanners from these algorithms is linear in n , it can still be rather large due to the hidden constants in the O -notation that depend on ε and the dimension d . Therefore there has also been attention to the problem with the priorities reversed: given a certain number of edges, how small a dilation can we realize? Das and Hefferman [3] showed how to compute in $O(n \log n)$ time, for any constant $\varepsilon' > 0$, a spanner with $(1 + \varepsilon')n$ edges, degree three, and constant dilation in the sense that it only depends on ε' and d . Aronov et al. [1] studied even smaller spanners and showed how to compute in $O(n \log n)$ time, for any constant $k > 0$, a spanner with $n - 1 + k$ edges and dilation $O(n/(k + 1))$. They proved that this is optimal in the worst case. With this work Aronov et al. provided a generalization (and better constants) of Eppstein's observation that any spanning tree on the vertices of a regular polygon has dilation $\Omega(n)$, and that the minimum-weight spanning tree achieves dilation $O(n)$ for any point set S [4]. Note that any spanner must at least connect the n points of S , and therefore include a spanning tree of S —spanners with fewer than $n - 1$ edges are impossible.

Eppstein and Aronov et al. guarantee that their spanning trees on n points have dilation within a constant factor of the best spanning tree on the worst-case set of n points. However, for a given set of points, it may be possible to achieve a much smaller dilation. The minimum-weight spanning tree is not always a good choice: Eppstein mentions an example where the minimum-weight spanning tree has dilation $\Theta(n)$ while dilation $\Theta(\sqrt{n})$ is possible. In Figure 1 we show an example where the minimum-weight spanning tree has dilation $\Theta(n)$ while dilation $\Theta(1)$ is possible.

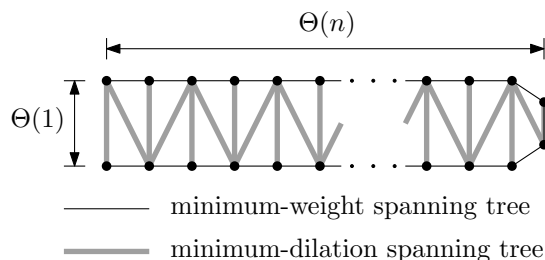


Figure 1: An example of a minimum-weight spanning tree with bad dilation.

A natural question arises: Given a set S of n points in \mathbb{R}^d , what is the spanning tree of S of minimum dilation? Eppstein posed the following questions:

Is it possible to construct the exact minimum-dilation geometric spanning tree, or an approximation to it, in polynomial time? Does the minimum-dilation spanning tree have any edge crossings?

Very recently these and related questions received the attention of Gudmundsson and Smid and of Klein and Kutz. Gudmundsson and Smid [6] found by reduction from 3SAT that, given a geometric graph G , a dilation δ and a number $k \geq n - 1$, it is NP-hard to decide whether G contains a δ -spanner with at most k edges. Klein and Kutz [7] are about to publish a proof that, given a set of points P in the plane, a dilation δ and a number $k \geq n - 1$, it is NP-hard to decide whether there is a plane δ -spanner with at most k edges. In both cases the proofs are based on instances of the problem with $k > n - 1$, and the question whether a spanning tree with dilation at most δ can be found in polynomial time remains open.

Klein and Kutz answer Eppstein's second question by providing a set of seven points whose minimum-dilation spanning tree has edge crossings.

Our results We first address Eppstein's second question: we show that even the minimum-dilation spanning tree of a set of only five points may have edge crossings. Sets of less than five points always have a minimum-dilation spanning tree without edge crossings.

We then give a negative answer to the first question: given a set S of n points in the plane and a dilation $\delta > 1$, it is NP-hard to decide whether a spanning tree of S with dilation at most δ exists—regardless if edge crossings are allowed or not. Thus the problems studied by Gudmundsson and Smid and by Klein and Kutz remain NP-hard even if the number of edges k is restricted to $n - 1$.

Our NP-hardness proof¹ is a reduction from PARTITION:

PARTITION

Given a sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of n positive integers, is there a partition of $\{1, \dots, n\}$ into subsets A and A' such that $A \cap A' = \emptyset$, $A \cup A' = \{1, \dots, n\}$ and $\sum_{i \in A} \alpha_i = \sum_{i \in A'} \alpha_i$?

We first show that a sequence of n positive integers can be transformed, in polynomial time, to $8n + 8$ points in the plane, such that a partition exists if and only if there exists a geometric spanning tree \mathcal{T} on S with $\Delta(\mathcal{T}) \leq 3/2$. Conceptually, our construction is quite simple, the difficulty being to ensure that no unwanted solutions or interactions can arise.

To prove NP-hardness of the problem, we have to do the reduction on a Turing-machine or an equivalent model, and so our spanning tree problem should take integer coordinates as input. Our construction does not quite achieve this: we construct points as the intersection of circles. We solve this problem by showing that if the coordinates of these points are approximated by rational points with precision polynomial in the input size, the construction still goes through, achieving a polynomial-time reduction to a problem of polynomial size.

¹Note that we cannot claim NP-completeness of the problem, as it is not known how to do the necessary distance computations involving sums of square roots in NP.

2 Minimum-dilation spanning trees with edge crossings

Suppose we are given a set S of points. Klein and Kutz [7] have an example where $|S| = 7$ and the minimum-dilation spanning tree of S has edge crossings. Below we give an example with $|S| = 5$, and prove that there is no smaller set S that does not have a crossing-free minimum-dilation spanning tree.

For $u, v \in S$, we call uv a δ -critical edge if for every point $w \in S \setminus \{u, v\}$ we have

$$\delta \cdot |uv| < |uw| + |vw|.$$

Clearly, any spanning tree \mathcal{T} of S that does not include all δ -critical edges has dilation $\Delta(\mathcal{T}) > \delta$.

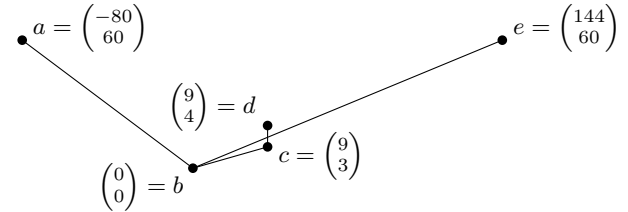


Figure 2: A set of five points whose minimum-dilation spanning tree has dilation $8/7$ and has an edge crossing (not to scale).

Figure 2 shows a set of five points $S = \{a, b, c, d, e\}$. The reader may verify that the edges ab , bc , and cd are $8/7$ -critical. To complete the spanning tree, it remains to add either ae , be , ce , or de to \mathcal{T} . Adding ae would make $d_{\mathcal{T}}(b, e)$ longer than $(8/7)|be|$, while choosing ce or de would make $d_{\mathcal{T}}(a, e)$ longer than $(8/7)|ae|$. On the other hand, including be results in $d_{\mathcal{T}}(a, e) = (8/7)|ae|$ and $\Delta(\mathcal{T}) = 8/7$. The minimum-dilation spanning tree of S thus consists of the edges ab , bc , cd and be , where cd and be intersect.

Theorem 1 For $n \geq 5$, there are sets of n points in the plane that do not have a minimum-dilation spanning tree without edge crossings. For $n < 5$, every set of n points in \mathbb{R}^d has a minimum-dilation spanning tree without edge crossings.

Proof. For $n = 5$, an example is given in Figure 2. The example can easily be extended with additional points.

For $n < 5$, observe that intersections between possible edges are possible only if $n = 4$ and the points are co-planar and in convex position. Suppose \mathcal{T} is a minimum-dilation spanning tree with an edge crossing on four such points a, b, c, d . Without loss of generality, assume ad and bc are the intersecting edges, cd is the third edge, and b lies closer to d than to c (see Figure 3). We now create another spanning tree \mathcal{T}' by taking \mathcal{T} and replacing edge bc by edge bd . This increases only $d_{\mathcal{T}}(b, c)$. Hence we get:

$$\begin{aligned} \Delta(\mathcal{T}') &= \max \left\{ \Delta(\mathcal{T}), \frac{d_{\mathcal{T}'}(b, c)}{|bc|} \right\} \\ &< \max \left\{ \Delta(\mathcal{T}), \frac{d_{\mathcal{T}}(b, d)}{|bd|} \right\} = \Delta(\mathcal{T}). \end{aligned}$$

So \mathcal{T}' is a minimum-dilation spanning tree of a, b, c and d without edge crossings. \square

Aronov et al. [1] already observed that minimum-dilation spanning paths may have edge crossings. Figure 4 shows an example. To get a spanning path of dilation at most $73/37$, we need to include

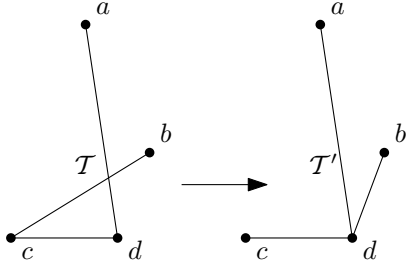


Figure 3: Any minimum-dilation spanning tree on four points that has an edge crossing can be transformed into a minimum-dilation spanning tree without any edge crossing.

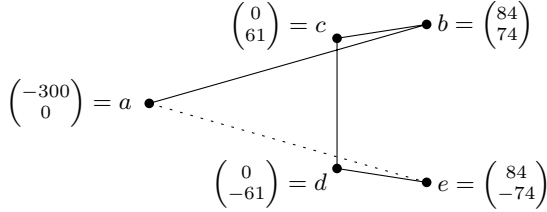


Figure 4: A set of points whose minimum-dilation spanning path and minimum-dilation spanning tour have dilation $73/37$ and have edge crossings (not to scale).

edges bc , cd and de , because these are all $73/37$ -critical. To complete the spanning path, we need to include ab (or its symmetric counterpart ae), which indeed yields a spanning path of dilation $73/37$ (where $d_{\mathcal{T}}(b, e) = (73/37)|be|$), and ab intersects cd . The unique minimum-dilation spanning tour of the same set of points is ab, bc, cd, de, ea and also has edge crossings.

3 Computing a minimum-dilation tree is NP-hard

For a set S of points in the plane, let us define $\Delta(S) := \min_{\mathcal{T}} \Delta(\mathcal{T})$, where the minimum is taken over all spanning trees \mathcal{T} of S .

Our NP-hardness proof is a reduction from PARTITION. The basic idea is simple: Given an instance of PARTITION, that is, a sequence of n positive integers, we construct a set S of $8n+8$ points in the plane such that $\Delta(S) \leq 3/2$ if and only if the partition problem has a solution.

In Section 3.1 we show how to construct this set S . In Section 3.2 we then show that if no partition exists, then $\Delta(S) > 3/2$. In Section 3.3 we show that if a partition exists, then $\Delta(S) \leq 3/2$, and there is a spanning tree with dilation $3/2$ on S that does not have any edge crossings. Finally we show in Section 3.4 that the entire construction can be done in such a way that the points of S have integer coordinates with total bit complexity polynomial in the bit complexity of the PARTITION instance. Together, we prove the following:

Theorem 2 *Given a set S of points with integer coordinates in the plane and two positive integers P and Q , it is NP-hard to decide whether a geometric spanning tree of S with dilation at most P/Q exists. The problem remains NP-hard if the spanning tree is restricted not to have edge crossings.*

3.1 Construction of S

We are given an instance of PARTITION, that is, a sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of n positive integers. We

define $\hat{\sigma} := \sum_{i=1}^n \hat{\alpha}_i$, and define the scaled quantities $\alpha_i = \hat{\alpha}_i / (10\hat{\sigma})$. By construction, we have $\sum_{i=1}^n \alpha_i = 1/10$.

Figure 5 shows the general structure of our construction of S . It is symmetric around the y -axis, and so we only need to describe the right half of the construction.

We create $3n+1$ points lying on the line with slope $3/4$ through the point $(5/2, 0)$:

$$\begin{aligned} a_i &= \binom{5/2}{0} + (4^{i-1} - 1) \binom{4}{3} & \forall 1 \leq i \leq n+1 \\ b_i &= a_i + \frac{4^{i-1}}{5} \binom{4}{3} & \forall 1 \leq i \leq n \\ c_i &= b_i + \frac{3 \cdot 4^{i-1}}{5} \binom{4}{3} & \forall 1 \leq i \leq n \end{aligned}$$

The distances between these points are as follows:

$$\begin{aligned} |a_i a_{i+1}| &= 15 \cdot 4^{i-1} \\ |a_i b_i| &= 1 \cdot 4^{i-1} \\ |b_i c_i| &= 3 \cdot 4^{i-1} \\ |c_i a_{i+1}| &= 11 \cdot 4^{i-1} \\ |a_1 a_{n+1}| &= 5(4^n - 1) \end{aligned}$$

So far, we haven't made any use of the quantities α_i . They appear in the definition of the n points d_i , for $1 \leq i \leq n$. These points lie slightly above the line $a_1 a_{n+1}$, and are defined by the two equations:

$$\begin{aligned} |d_i a_{i+1}| &= 2 \cdot 4^{i-1} \\ |c_i d_i| &= 9 \cdot 4^{i-1} + \alpha_i \end{aligned}$$

Figure 6 shows the interval between a_i and a_{i+1} .

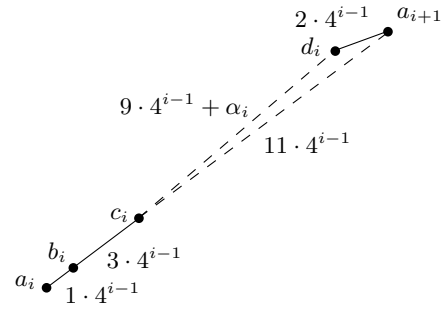


Figure 6: The construction between a_i and a_{i+1}

Since $|c_i a_{i+1}| = 11 \cdot 4^{i-1}$ and $0 < \alpha_i \leq 1/10$, it is clear that d_i exists. We add two more points at the far end:

$$\begin{aligned} p_1 &= a_{n+1} + \left(\frac{4^n}{9} - \frac{179}{1800}\right) \binom{3}{-4} \\ p_2 &= a_{n+1} + 4\left(\frac{4^n}{9} - \frac{179}{1800}\right) \binom{3}{-4} \end{aligned}$$

Both points lie on the line through a_{n+1} with slope $-4/3$, and so $\angle a_1 a_{n+1} p_2$ is a right angle. We have

$$\begin{aligned} |a_{n+1} p_1| &= \frac{5}{9} 4^n - \frac{179}{360} \\ |a_{n+1} p_2| &= 4|a_{n+1} p_1| = \frac{5}{9} 4^{n+1} - \frac{179}{90} \end{aligned}$$

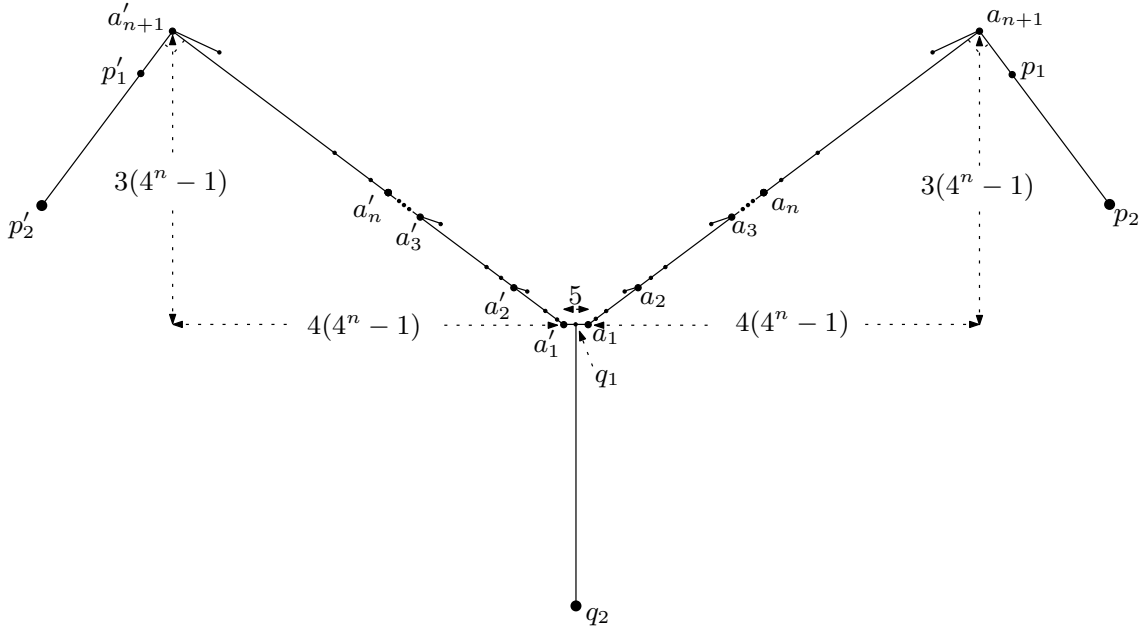


Figure 5: Construction of S

We denote the mirror images under reflection in the y -axis of the $4n + 3$ points a_i, b_i, c_i, d_i, p_i constructed so far as $a'_i, b'_i, c'_i, d'_i, p'_i$. Our point set S consists of $8n + 8$ points, namely the original points, their mirror images, and two more points on the y -axis:

$$q_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$q_2 = \begin{pmatrix} 0 \\ -\frac{25}{9}4^n + \frac{11}{18} \end{pmatrix}$$

We have

$$p_2 - q_2 = \left(\frac{4^{n+1}}{3} - \frac{101}{150} \right) \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

so q_2p_2 is parallel to a_1a_{n+1} , and

$$|q_2p_2| = |q_2p'_2| = \frac{5}{3} \cdot 4^{n+1} - \frac{101}{30}.$$

We now prove some basic properties of the constructed point set S .

Lemma 1 *We have $\cos \angle c_i a_{i+1} d_i > 1 - 4^{1-i}/22 \geq 21/22$, and the y -coordinate of d_i is strictly smaller than the y -coordinate of a_{i+1} , for $1 \leq i \leq n$.*

Proof. Since $\alpha_i \leq \frac{1}{10}$, the cosine theorem gives

$$\begin{aligned} \cos \angle c_i a_{i+1} d_i &= \frac{|c_i a_{i+1}|^2 + |d_i a_{i+1}|^2 - |c_i d_i|^2}{2 \cdot |c_i a_{i+1}| \cdot |d_i a_{i+1}|} \\ &\geq \frac{11^2 + 2^2 - (9 + 4^{1-i}/10)^2}{2 \cdot 11 \cdot 2} \\ &> 1 - 4^{1-i}/22 \geq 21/22 > 4/5, \end{aligned}$$

and so $\angle c_i a_{i+1} d_i$ is smaller than the angle of $a_1 a_{n+1}$ with the horizontal. \square

Corollary 1 *The cosine of the angle of $|c_i d_i|$ with the horizontal is more than $(\frac{4}{5} \cdot 11 - 2)/(9 + 4^{1-i} \alpha_i) \geq 68/91$.*

For $u, v \in S$, we call uv a *critical edge* if for every $w \in S \setminus \{u, v\}$ we have

$$1.6|uv| < |uw| + |wv|.$$

As we observed in the previous section, any spanning tree \mathcal{T} on S that does not include a critical edge uv must have dilation $\Delta(\mathcal{T}) > 1.6$. Let us call the point $w \in S \setminus \{u, v\}$ minimizing the sum $|uw| + |wv|$ the *nearest neighbor* of uv .

Lemma 2 *The following edges are all critical: $q_1 a_1, a_{n+1} p_1, p_1 p_2, a_i b_i, b_i c_i, d_i a_{i+1}$ (where $1 \leq i \leq n$), and their mirror images.*

Proof. The nearest neighbor of $q_1 a_1$ is b_1 . The edge $q_1 a_1$ is critical since

$$|q_1 b_1| + |b_1 a_1| = \sqrt{3.3^2 + 0.6^2} + 1 > 1.6 \cdot \frac{5}{2}.$$

The nearest neighbor of $a_{n+1} p_1$ is d_n . Since $\angle d_n a_{n+1} p_1$ is obtuse, we have

$$\begin{aligned} |a_{n+1} d_n| + |d_n p_1| &\geq 2 \cdot 4^{n-1} + \frac{5}{9}4^n - \frac{179}{360} = \\ &= \frac{19}{18}4^n - \frac{179}{360} > 1.6 \left(\frac{5}{9}4^n - \frac{179}{360} \right) = 1.6|a_{n+1} p_1|, \end{aligned}$$

and so $a_{n+1} p_1$ is critical.

The nearest neighbor of $p_1 p_2$ is a_{n+1} . The edge is critical since

$$|p_1 a_{n+1}| + |a_{n+1} p_2| = \frac{5}{3}|p_1 p_2| > 1.6|p_1 p_2|.$$

The edge $a_1 b_1$ is critical since its nearest neighbor is q_1 and

$$|a_1 q_1| + |q_1 b_1| > 5 > 1.6|a_1 b_1|.$$

For $2 \leq i \leq n$, the nearest neighbor of $a_i b_i$ is d_{i-1} . By Lemma 1, the y -coordinate of d_{i-1} is strictly smaller than the y -coordinate of a_i , so $\cos \angle d_{i-1} a_i b_i < 0$, which bounds $|d_{i-1} b_i|^2 > |d_{i-1} a_i|^2 + |a_i b_i|^2 = (\frac{\sqrt{5}}{2} \cdot 4^{i-1})^2$. We get

$$|a_i d_{i-1}| + |d_{i-1} b_i| > 2 \cdot 4^{i-2} + \frac{\sqrt{5}}{2} \cdot 4^{i-1} > 1.6|a_i b_i|,$$

and so $a_i b_i$ is critical.

The nearest neighbor of $b_i c_i$ is a_i , and the edge is critical since

$$|b_i a_i| + |a_i c_i| = 5 \cdot 4^{i-1} > 1.6|b_i c_i|.$$

For $1 \leq i \leq n-1$, the nearest neighbor of $d_i a_{i+1}$ is b_{i+1} , and the edge is critical since

$$|d_i b_{i+1}| + |b_{i+1} a_{i+1}| > 2|b_{i+1} a_{i+1}| = 2 \cdot 4^i > 1.6|d_i a_{i+1}|.$$

Finally, the nearest neighbor of $d_n a_{n+1}$ is p_1 , and

$$\begin{aligned} |d_n p_1| + |p_1 a_{n+1}| &> 2|p_1 a_{n+1}| = \\ &= 2\left(\frac{5}{9}4^n - \frac{179}{360}\right) > 1.6(2 \cdot 4^{n-1}) = 1.6|d_n a_{n+1}| \end{aligned}$$

implies that $d_n a_{n+1}$ is critical. \square

The enumeration in Lemma 2 is exhaustive: these are all the critical edges. However, to form the connection between c_i and d_i , only two choices are possible—this is the choice at the heart of our NP-hardness argument.

Lemma 3 *If \mathcal{T} is a spanning tree on S with $\Delta(\mathcal{T}) \leq 1.6$, then it contains exactly one of the edges $c_i d_i$ and $c_i a_{i+1}$, and exactly one of the edges $c'_i d'_i$ and $c'_i a'_{i+1}$, for each $1 \leq i \leq n$.*

Proof. Consider points $c_i d_i$, for some $1 \leq i \leq n$. If \mathcal{T} contains neither $c_i d_i$ nor $c_i a_{i+1}$, then the shortest path from c_i to d_i in \mathcal{T} must make use of a point $w \in S \setminus \{c_i, d_i, a_{i+1}\}$, and its length is at least $|c_i w| + |w d_i|$. The point w minimizing this expression is b_i , but since

$$|d_i b_i| + |b_i c_i| > (11 + 3 - 2)4^{i-1} + 3 \cdot 4^{i-1} > 1.6|c_i d_i|,$$

this is not good enough. It follows that \mathcal{T} must contain at least one of the edges $c_i d_i$ or $c_i a_{i+1}$. Since by Lemma 2 it also contains $d_i a_{i+1}$, it cannot contain both edges. \square

3.2 If there is no partition, then $\Delta(S) > 3/2$

In fact, we will prove a slightly stronger claim: If there is no solution to the PARTITION instance, then $\Delta(S) > 3/2 + \xi$, where $\xi := 1/(4^{n+4}\sigma)$. Throughout this section, we will assume that a spanning tree \mathcal{T} on S exists with $\Delta(\mathcal{T}) \leq 3/2 + \xi$. We define

$$\begin{aligned} A &:= \{i \in \{1, \dots, n\} \mid \mathcal{T} \text{ contains } c_i d_i\}, \\ A' &:= \{i \in \{1, \dots, n\} \mid \mathcal{T} \text{ contains } c'_i d'_i\}, \end{aligned}$$

and our aim is to show that A, A' are a solution to the PARTITION instance.

We set $\sigma_A = \sum_{i \in A} \alpha_i$ and $\sigma_{A'} = \sum_{i \in A'} \alpha_i$. We need to show that $\sigma_A = \sigma_{A'}$, that $A \cap A' = \emptyset$, and that $A \cup A' = \{1, \dots, n\}$.

Lemma 4 *We have $A \cup A' = \{1, \dots, n\}$.*

Proof. Let us assume that for some $1 \leq i \leq n$, neither $c_i d_i$ nor $c'_i d'_i$ is in \mathcal{T} . We consider the dilation of the pair $d'_i d_i$. The shortest path from d_i to d'_i in \mathcal{T} must go through both a_{i+1} and a'_{i+1} , and so its length is at least

$$\begin{aligned} d_{\mathcal{T}}(d'_i, d_i) &\geq 2|d_i a_{i+1}| + 2|a_1 a_{i+1}| + |a_1 a'_1| \\ &= 4 \cdot 4^{i-1} + 10(4^i - 1) + 5 \\ &= 11 \cdot 4^i - 5. \end{aligned}$$

On the other hand, $|d'_i d_i| = |a'_{i+1} a_{i+1}| - 2\ell$, where ℓ is the length of the projection of $d_i a_{i+1}$ on the x -axis. By Lemma 1, we have $\ell > \frac{4}{5}|d_i a_{i+1}|$, and so

$$\begin{aligned} |d'_i d_i| &= |a'_{i+1} a_{i+1}| - 2\ell \\ &< |a'_{i+1} a_{i+1}| - \frac{8}{5}|d_i a_{i+1}| \\ &= 8(4^i - 1) + 5 - \frac{16}{5}4^{i-1} \\ &= 7.2 \cdot 4^i - 3, \end{aligned}$$

and so

$$\begin{aligned} d_{\mathcal{T}}(d'_i, d_i)/|d'_i d_i| &> (11 \cdot 4^i - 5)/(7.2 \cdot 4^i - 3) \\ &> 3/2 + 1/4^4 \geq 3/2 + \xi. \end{aligned}$$

This is a contradiction, so no such i can exist, and the lemma follows. \square

Lemma 5 *We have $A \cap A' = \emptyset$ and $\sigma_A = \sigma_{A'} = 1/20$. Also, \mathcal{T} contains the edge $q_1 q_2$.*

Proof. The spanning tree \mathcal{T} must contain the $6n + 6$ critical edges enumerated in Lemma 2 since $1.6 > 3/2 + \xi$. By Lemma 3, it must also contain n edges connecting each c_i to either d_i or a_{i+1} , and by symmetry also n edges connecting each c'_i to either d'_i or a'_{i+1} . Since S consists of $8n + 8$ points, \mathcal{T} has $8n + 7$ edges, and so there is only one edge unaccounted for. This edge must connect q_2 to some point $q \in S \setminus \{q_2\}$. We note that $|q_2 q| \geq |q_2 q_1| = \frac{25}{9}4^n - \frac{11}{18}$ (see Figure 5).

Since $\Delta(\mathcal{T}) \leq 3/2 + \xi$, we have

$$\begin{aligned} d_{\mathcal{T}}(p'_2, q_2) + d_{\mathcal{T}}(q_2, p_2) &\leq \frac{3}{2}(|p'_2 q_2| + |q_2 p_2|) + \xi(|p'_2 q_2| + |q_2 p_2|) \\ &< 5 \cdot 4^{n+1} - \frac{101}{10} + \frac{1}{10\sigma}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d_{\mathcal{T}}(p'_2, q_2) + d_{\mathcal{T}}(q_2, p_2) &= d_{\mathcal{T}}(p'_2, q) + |qq_2| + |q_2 q| + d_{\mathcal{T}}(q, p_2) \\ &\geq d_{\mathcal{T}}(p'_2, p_2) + 2|qq_2| \\ &\geq d_{\mathcal{T}}(p'_2, p_2) + \frac{50}{9}4^n - \frac{11}{9}. \end{aligned}$$

Now,

$$\begin{aligned} d_{\mathcal{T}}(p'_2, p_2) &= |p'_2 a'_{n+1}| + d_{\mathcal{T}}(a'_{n+1}, a'_1) \\ &\quad + |a'_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &= \frac{10}{9}4^{n+1} + \frac{46}{45} + d_{\mathcal{T}}(a'_1, a'_{n+1}) + d_{\mathcal{T}}(a_1, a_{n+1}) \end{aligned}$$

What is $d_{\mathcal{T}}(a_1, a_{n+1})$? Since the shortest path from a_1 to a_{n+1} in \mathcal{T} must go through all a_i , we can express it as

$$d_{\mathcal{T}}(a_1, a_{n+1}) = \sum_{i=1}^n d_{\mathcal{T}}(a_i, a_{i+1})$$

We now observe that $d_{\mathcal{T}}(a_i, a_{i+1}) = |a_i a_{i+1}|$ if \mathcal{T} contains $c_i a_{i+1}$, that is if $i \notin A$, and $d_{\mathcal{T}}(a_i, a_{i+1}) = |a_i a_{i+1}| + \alpha_i$ if $i \in A$. This implies

$$d_{\mathcal{T}}(a_1, a_{n+1}) = |a_1 a_{n+1}| + \sum_{i \in A} \alpha_i = 5(4^n - 1) + \sigma_A,$$

and similarly we have $d_{\mathcal{T}}(a'_1, a'_{n+1}) = 5(4^n - 1) + \sigma_{A'}$.

This gives

$$d_{\mathcal{T}}(p'_2, p_2) = \frac{130}{9}4^n - \frac{404}{45} + \sigma_A + \sigma_{A'}.$$

Putting everything together we get

$$\begin{aligned} 5 \cdot 4^{n+1} - \frac{101}{10} + \frac{1}{10\dot{\sigma}} &> d_{\mathcal{T}}(p'_2, q_2) + d_{\mathcal{T}}(q_2, p_2) \\ &\geq d_{\mathcal{T}}(p'_2, p_2) + \frac{50}{9}4^n - \frac{11}{9} \\ &= 5 \cdot 4^{n+1} - \frac{102}{10} + \sigma_A + \sigma_{A'}, \end{aligned}$$

which implies $\sigma_A + \sigma_{A'} < 1/10 + 1/(10\dot{\sigma})$.

If there is an $i \in A \cap A'$, then Lemma 4 implies $\sigma_A + \sigma_{A'} \geq 1/10 + \alpha_i = 1/10 + \dot{\alpha}_i/(10\dot{\sigma})$. Since $\dot{\alpha}_i$ is a positive integer, this is a contradiction, and so $A \cap A' = \emptyset$ and $\sigma_A + \sigma_{A'} = 1/10$.

We now show that only $q = q_1$ is possible. We use again

$$\begin{aligned} d_{\mathcal{T}}(q_2, p_2) &\leq \frac{3}{2}|q_2 p_2| + \xi|q_2 p_2| \\ &< 10 \cdot 4^n - \frac{101}{20} + \frac{1}{20\dot{\sigma}} \\ &\leq 10 \cdot 4^n - \frac{100}{20}. \end{aligned}$$

If q is on the left side of the y -axis, then the path from q_2 to p_2 in \mathcal{T} passes through a'_1 , and we have

$$\begin{aligned} d_{\mathcal{T}}(q_2, p_2) &\geq |q_2 q| + |a'_1 q_1| + d_{\mathcal{T}}(q_1, p_2) \\ &\geq |q_2 q_1| + 2|q_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &\geq 10 \cdot 4^n - \frac{52}{20}, \end{aligned}$$

a contradiction. Similarly, q cannot be on the right side of the y -axis, and the only remaining possibility is $q = q_1$.

It remains to show that $\sigma_A = \sigma_{A'} = 1/20$. If this is not the case, we can without loss of generality assume $\sigma_A > 1/20$. Since $\sum_{i \in A} \dot{\alpha}_i - \dot{\sigma}/2 > 0$ is an integer, we have $\sum_{i \in A} \dot{\alpha}_i - \dot{\sigma}/2 \geq 1$, and so $\sigma_A \geq 1/20 + 1/(10\dot{\sigma})$. On the other hand, we have

$$\begin{aligned} 10 \cdot 4^n - \frac{101}{20} + \frac{1}{20\dot{\sigma}} &> d_{\mathcal{T}}(q_2, p_2) \\ &= |q_2 q_1| + |q_1 a_1| + d_{\mathcal{T}}(a_1, a_{n+1}) + |a_{n+1} p_2| \\ &= 10 \cdot 4^n - \frac{102}{20} + \sigma_A, \end{aligned}$$

and so $\sigma_A < 1/20 + 1/(20\dot{\sigma})$, a contradiction. \square

3.3 If a set partition exists, then $\Delta(S) \leq 3/2$

Let us call a tree \mathcal{T} on S a *standard tree* if it consists of the critical edges, the edge $q_1 q_2$, and for each $1 \leq i \leq n$ either $c_i d_i$ or $c_i a_{i+1}$ and either $c'_i d'_i$ or $c'_i a'_{i+1}$. In the following lemmas we will show that *any* standard tree has dilation less than $3/2$ for nearly all pairs of points in S , excluding only the pairs (d'_i, d_i) (for $1 \leq i \leq n$), (q_2, p_2) , and (q_2, p'_2) . These remaining pairs are where the existence of a solution to the PARTITION problem is critical.

Let \mathcal{T} be an arbitrary standard tree. Let H be the set of points of S to the right of the y -axis, except p_1 and p_2 . Symmetrically, let H' be the set of points of S to the left of the y -axis, except p'_1 and p'_2 .

$$H := \{a_i, b_j, c_j, d_j \mid 1 \leq i \leq n+1, 1 \leq j \leq n\}$$

$$H' := \{a'_i, b'_j, c'_j, d'_j \mid 1 \leq i \leq n+1, 1 \leq j \leq n\}$$

Below, in Lemma 6 and 7 we first prove that the dilation on paths within $H \cup \{q_1\}$ is less than $3/2$. By

symmetry, these lemmas also apply to paths within $H' \cup \{q_1\}$. Next, in Lemma 8, 9, and 10, we analyse the dilation on paths between H and H' , except paths from d'_i to d_i (for $1 \leq i \leq n$) such that \mathcal{T} contains neither $c_i d_i$ nor $c'_i d'_i$. Lemma 11 deals with paths from $\{p'_2, p'_1, p_1, p_2\}$ to $\{p'_2, p'_1\} \cup H' \cup \{q_1\} \cup H \cup \{p_1, p_2\} = S \setminus \{q_2\}$. It remains to consider the dilation on pairs that involve $\{q_2\}$: Lemma 12 treats this case, except for the pairs (q_2, p_2) and (q_2, p'_2) . We then show in Lemma 13 that if a solution to the PARTITION problem exists, we can get dilation at most $3/2$ also on (q_2, p_2) , (q_2, p'_2) and on all pairs (d'_i, d_i) (for $1 \leq i \leq n$). Thus we prove that if a PARTITION solution exists, $\Delta(S) \leq 3/2$.

To facilitate the analysis, we use the following notation: denote by w^\downarrow the orthogonal projection of w on the line through a_1 and a_{n+1} . Let $P_{\mathcal{T}}(u, v)$ be the path from u to v in \mathcal{T} . The edges and vertices of the path may depend on the choice of \mathcal{T} : for example, d_i lies on $P_{\mathcal{T}}(a_1, a_{n+1})$ if and only if \mathcal{T} contains $c_i d_i$.

We first concentrate on the dilation between points a_i, b_i, c_i and d_i in one half of the tree.

Lemma 6 *For any pair of points (not necessarily vertices) $u, v \in P_{\mathcal{T}}(a_{n+1}, w)$, where $w \in H$, we have $d_{\mathcal{T}}(u, v) < (22/21)|u^\downarrow v^\downarrow|$ and $\Delta_{\mathcal{T}}(u, v) < 22/21 < 3/2$.*

Proof. By Lemma 1 the cosine of the angle between any segment of the path $P_{\mathcal{T}}(a_1, a_{n+1})$ and the line $a_1 a_{n+1}$ is more than $21/22$; hence the path is monotone in its projection on the line $a_1 a_{n+1}$ and each segment has length at most $22/21$ times the length of its projection. Since $|u^\downarrow v^\downarrow| \leq |uv|$, it follows that $\Delta_{\mathcal{T}}(u, v) < 22/21$. \square

Lemma 7 *For any pair of vertices $u, v \in H \cup \{q_1\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.*

Proof. We first deal with the case of $u, v \in H$. Without loss of generality, let u lie above and to the right of v . If u lies on the path $P_{\mathcal{T}}(v, a_{n+1})$, the lemma follows from Lemma 6. Otherwise, $u = d_i$ for some $1 \leq i \leq n$, and \mathcal{T} does not contain the edge $c_i d_i$. Now we have:

$$\begin{aligned} \frac{d_{\mathcal{T}}(d_i, v)}{|d_i v|} &\leq \frac{d_{\mathcal{T}}(d_i, d'_i) + d_{\mathcal{T}}(d'_i, v)}{|d_i^\downarrow v^\downarrow|} \\ &\leq \frac{d_{\mathcal{T}}(d_i, d'_i)}{|d_i^\downarrow c_i|} + \frac{d_{\mathcal{T}}(d'_i, v)}{|d_i^\downarrow v^\downarrow|} \\ &< 4/9 + 22/21 = 94/63 < 3/2. \end{aligned}$$

This concludes the proof for the case of $u, v \in H$. Now suppose $v = q_1$. If $u = a_1, b_1, c_1$ or d_1 , it can easily be verified that $\Delta_{\mathcal{T}}(u, q_1) < 3/2$ (regardless whether the path $P_{\mathcal{T}}(d_1, q_1)$ passes through a_2). If u is any other point in H , then the path $P_{\mathcal{T}}(u, q_1)$ passes through a_2 . Now observe:

$$d_{\mathcal{T}}(a_2, q_1) \leq 17.5 + \sigma < \frac{22}{21} \cdot 17 = \frac{22}{21}|a_2, q_1^\downarrow|.$$

Hence we can apply the same arguments as for $u, v \in H$ to bound the dilation of the path $P_{\mathcal{T}}(u, q_1)$ to less than $3/2$. \square

In the following three lemmas we turn our attention to pairs of points in opposite halves of the tree (still excluding p_1, p_2, p'_1, p'_2 and q_2).

Lemma 8 *For any pair of points (not necessarily vertices) $u, v \in P_{\mathcal{T}}(a_{n+1}, a'_{n+1})$, we have $\Delta_{\mathcal{T}}(u, v) < 91/68 < 3/2$.*

Proof. By Corollary 1, the cosine of the angle of any segment of $P_{\mathcal{T}}(a_{n+1}, a'_{n+1})$ and the x -axis is more than $68/91$. Hence the path is x -monotone and its dilation is less than $91/68$. \square

To facilitate the analysis of the dilation of pairs that involve a point d_i or d'_i , we introduce an auxiliary point d_i^* on $a_i a_{i+1}$:

$$d_i^* = c_i + \frac{9 \cdot 4^{i-1}}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = a_{i+1} - \frac{2 \cdot 4^{i-1}}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

and similarly define $d_i^{*'} on $a'_i a'_{i+1}$. We have $|d_i^* a_{i+1}| = |d_i a_{i+1}| = 2 \cdot 4^{i-1}$, see Figure 7. Since by Lemma 1$

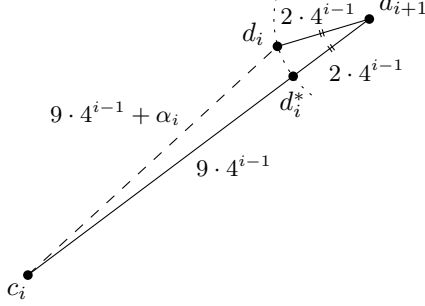


Figure 7: The point d_i^*

we have $\cos \angle c_i a_{i+1} d_i > 1 - 4^{1-i}/22$, we can use the cosine theorem to bound $|d_i d_i^*|$:

$$\begin{aligned} |d_i d_i^*|^2 &= |d_i a_{i+1}|^2 + |d_i^* a_{i+1}|^2 \\ &\quad - 2|d_i a_{i+1}||d_i^* a_{i+1}| \cos \angle c_i a_{i+1} d_i \\ &= 2(2 \cdot 4^{i-1})^2 (1 - \cos \angle c_i a_{i+1} d_i) \\ &< 2(2 \cdot 4^{i-1})^2 \frac{1}{22 \cdot 4^{i-1}} = \frac{4^i}{11} \end{aligned}$$

and so

$$|d_i d_i^*| < \sqrt{4^i/11} \quad (1)$$

Lemma 9 For any pair of points (d'_i, u) , where $1 \leq i \leq n$ and u is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a_{n+1}, a_1)$, we have $\Delta_{\mathcal{T}}(d'_i, u) < 3/2$.

Proof. If d'_i lies on the path $P_{\mathcal{T}}(a'_1, a'_{n+1})$, the lemma follows from Lemma 8. Otherwise, \mathcal{T} contains $c'_i a'_{i+1}$ (and not $c'_i d'_i$).

The ratio $|uu^\perp|/|a_1 u^\perp|$ is maximized for $u = d_j$, for some j , so with Equation 1 we get:

$$\begin{aligned} \frac{|uu^\perp|}{|a_1 u^\perp|} &= \max_j \frac{|d_j d_j^\perp|}{|a_1 d_j^\perp|} < \max_j \frac{|d_j d_j^*|}{|a_1 d_j^*|} \\ &< \max_j \frac{2^j/\sqrt{11}}{5(\frac{9}{10}4^j - 1)} < 1/20 \end{aligned}$$

We set $m' = \frac{9}{10}4^i - 1$ and $m = |a_1 u^\perp|/5$, and have

$$\begin{aligned} d_i^{*'} &= a'_1 + m' \begin{pmatrix} -4 \\ 3 \end{pmatrix} \\ u^\perp &= a_1 + m \begin{pmatrix} 4 \\ 3 \end{pmatrix} \end{aligned}$$

and thus:

$$\begin{aligned} |d_i^{*'} d_i^*| &\leq \frac{1}{20} \cdot 5m' = \frac{1}{4}m' \\ |uu^\perp| &\leq \frac{1}{20} \cdot 5m = \frac{1}{4}m \end{aligned}$$

We can now bound $d_{\mathcal{T}}(u, d'_i)$:

$$\begin{aligned} d_{\mathcal{T}}(u, d'_i) &\leq d_{\mathcal{T}}(u, a_1) + |a_1 a'_1| \\ &\quad + d_{\mathcal{T}}(a'_1, d_i^{*'}) + d_{\mathcal{T}}(d_i^{*'}, d'_i) \\ &\leq (|u^\perp a_1| + \sigma) + 5 \\ &\quad + (|a'_1 d_i^{*'}| + \sigma) + 2|a'_{i+1} d'_i| \\ &= 5m + 1/10 + 5 + 5m' + 1/10 + 4^i \\ &= 5m + 5m' + \frac{10}{9}(m' + 1) + 26/5 \\ &= 5m + \frac{55}{9}m' + \frac{284}{45} \end{aligned}$$

On the other hand,

$$\begin{aligned} |ud_i| &\geq |u^\perp d_i^{*'}| - |u^\perp u| - |d'_i d_i^{*'}| \\ &> |u^\perp d_i^{*'}| - \frac{1}{4}(m + m') \end{aligned}$$

It remains to show that $d_{\mathcal{T}}(u, d_i) \leq \frac{3}{2}|ud_i|$. This follows from:

$$10m + \frac{110}{9}m' + \frac{568}{45} \leq 3|u^\perp d_i^*| - \frac{3}{4}(m + m')$$

which follows from:

$$\begin{aligned} &((10 + \frac{3}{4})m + (\frac{110}{9} + \frac{3}{4})m' + \frac{568}{45})^2 \\ &< (11m + 13m' + 13)^2 + 4(\sqrt{26}m - \sqrt{14}m')^2 \\ &= 225m^2 + 225m'^2 + (286 - 8\sqrt{364})mm' \\ &\quad + 286m + 338m' + 169 \\ &< 225m^2 + 225m'^2 + 136mm' + 360m + 360m' + 225 \\ &= 9(4m + 5 + 4m')^2 + 9(3m - 3m')^2 \\ &= (3|u^\perp d_i^{*'}|)^2, \end{aligned}$$

completing the proof. \square

Note that the above Lemma applies symmetrically to pairs of points (d_i, u) where $1 \leq i \leq n$ and u is a point (not necessarily a vertex) on $P_{\mathcal{T}}(a'_{n+1}, a'_1)$.

Lemma 10 For any pair of vertices d'_i, d_j with $1 \leq i, j \leq n$ and $i \neq j$, we have $\Delta_{\mathcal{T}}(d'_i, d_j) < 3/2$.

Proof. If d'_i is on the path from a'_{n+1} to a'_1 , or if d_j is on the path from a_{n+1} to a_1 , the Lemma follows from Lemma 9.

Otherwise, \mathcal{T} contains $c'_i a'_{i+1}$ (not $c'_i d'_i$) and $c'_j a'_{j+1}$ (not $c'_j d_j$). Without loss of generality, assume that $i < j$. We set $m' = \frac{9}{10}4^i - 1$ and $m = \frac{9}{10}4^j - 1$, and have:

$$m + 1 = 4^{j-i}(m' + 1). \quad (2)$$

By equation (1) we have:

$$\begin{aligned} |d'_i d_i^{*'}| &< \sqrt{\frac{4^i}{11}} = \sqrt{\frac{10}{99}}(m' + 1) < \frac{1}{3}\sqrt{m' + 1} \\ |d_j d_j^*| &< \sqrt{\frac{4^j}{11}} = \sqrt{\frac{10}{99}}(m + 1) < \frac{1}{3}\sqrt{m + 1} \end{aligned}$$

We now bound $d_{\mathcal{T}}(d'_i, d_j)$:

$$\begin{aligned} d_{\mathcal{T}}(d'_i, d_j) &\leq |d'_i a'_{i+1}| + (|a'_{i+1} a'_1| + \sigma) + |a'_1 a_1| \\ &\quad + (|a_1 a_{j+1}| + \sigma) + |a_{j+1} d_j| \\ &= \frac{5}{9}(m' + 1) + (\frac{50}{9}m' + \frac{5}{9} + \frac{1}{10}) + 5 \\ &\quad + (\frac{50}{9}m + \frac{5}{9} + \frac{1}{10}) + \frac{5}{9}(m + 1) \\ &= \frac{55}{9}(m' + m) + \frac{334}{45} \end{aligned}$$

With equation (2) we now get:

$$d_{\mathcal{T}}(d'_i, d_j) < \frac{55}{9}(4^{j-i} + 1)(m' + 1)$$

On the other hand

$$\begin{aligned} |d'_i d_j| &\geq |d_i^* d_j^*| - |d_i^* d_i'^*| - |d_j^* d_j'| \\ &> \sqrt{(4m' + 5 + 4m)^2 + (3m - 3m')^2} \\ &\quad - \frac{1}{3}\sqrt{m' + 1} - \frac{1}{3}\sqrt{m + 1} \\ &> \sqrt{16(m + m')^2 + 9(m - m')^2} \\ &\quad - \frac{1}{3}\sqrt{m' + 1} - \frac{1}{3}\sqrt{m + 1} \end{aligned}$$

For bounding $d_{\mathcal{T}}(d'_i, d_j)/|d'_i d_j|$ we now consider two cases: $j = i + 1$, and $j > i + 1$. We first consider the case $j = i + 1$. By equation (2) we now have $m = 4m' + 3$, and thus:

$$\begin{aligned} |d'_i d_j| &> \sqrt{16(m + m')^2 + 9(m - m')^2} \\ &\quad - \frac{1}{3}\sqrt{m' + 1} - \frac{1}{3}\sqrt{m + 1} \\ &= \sqrt{16(5m' + 3)^2 + 9(3m' + 3)^2} \\ &\quad - \sqrt{m' + 1} \\ &= \sqrt{481m'^2 + 642m' + 225} - \sqrt{m' + 1} \\ &> (\sqrt{481} - 1)(m' + 1) \end{aligned}$$

Hence:

$$\frac{d_{\mathcal{T}}(d'_i, d_j)}{|d'_i, d_j|} < \frac{55}{9}(4^{j-i} + 1) < \frac{275/9}{\sqrt{481} - 1} < \frac{3}{188/9} < \frac{3}{2}$$

It remains to consider the case where $j > i + 1$. By equation (2) we have $m + m' > m - m' = (4^{j-i} - 1)(m' + 1)$. Thus we get:

$$\begin{aligned} |d'_i d_j| &> \sqrt{16(m + m')^2 + 9(m - m')^2} \\ &\quad - \frac{1}{3}\sqrt{m' + 1} - \frac{1}{3}\sqrt{m + 1} \\ &> 5(m - m') - \frac{1}{3}(m' + 1 + m + 1) \\ &= 5(4^{j-i} - 1)(m' + 1) - \frac{1}{3}(4^{j-i} + 1)(m' + 1) \\ &= \frac{14}{3}(4^{j-i} + 1)(m' + 1) - 10(m' + 1) \end{aligned}$$

Hence:

$$\begin{aligned} \frac{d_{\mathcal{T}}(d'_i, d_j)}{|d'_i, d_j|} &< \frac{55}{9}(4^{j-i} + 1) \\ &= \frac{55/3}{14 - 30/(4^{j-i} + 1)} \leq \frac{55/3}{208/17} = \frac{935}{624} < \frac{3}{2}. \end{aligned}$$

□

We now study pairs of vertices involving p_1, p_2, p'_1 and/or p'_2 , but not q_2 .

Lemma 11 For any pair of vertices u, v where $u \in \{p_1, p_2, p'_1, p'_2\}$ and $v \in S \setminus \{q_2\}$, we have $\Delta_{\mathcal{T}}(u, v) < 3/2$.

Proof. We assume that $u \in \{p_1, p_2\}$ (the case of $u \in \{p'_1, p'_2\}$ is symmetric). We now distinguish four cases for v : first $v \in \{p_1, p_2, a_{n+1}\}$, second $v \in H \setminus \{a_{n+1}\}$, third $v \in H' \cup \{q_1\}$, and fourth $v \in \{p'_1, p'_2\}$.

First, if $v \in \{p_1, p_2, a_{n+1}\}$, then the connection between u and v is a straight line and the dilation is 1.

Second, if $v \in H \setminus \{a_{n+1}\}$, then the path from u to v goes through a_{n+1} , and $\angle u a_{n+1} v \geq \pi/2$. Hence the dilation for the pair (u, v) is (using Lemma 6):

$$\frac{d_{\mathcal{T}}(u, v)}{|uv|} < \frac{|ua_{n+1}| + \frac{22}{21}|a_{n+1}v|}{(|ua_{n+1}| + |a_{n+1}v|)/\sqrt{2}} < \frac{22}{21}\sqrt{2} < \frac{3}{2}.$$

Third, if $v \in H' \cup \{q_1\}$, let w be a point where the segment uv intersects the path from a_{n+1} to a_1 (which is a part of the path from u to v). By the analysis of the previous case $\Delta_{\mathcal{T}}(u, w) < 3/2$, and by Lemma 8 or 9 we have $\Delta_{\mathcal{T}}(w, v) < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$.

Finally, if $v \in \{p'_1, p'_2\}$, let w be defined as above, and let w' be a point where the segment uv intersects the path from a'_1 to a'_{n+1} . By the analysis of the second case $\Delta_{\mathcal{T}}(u, w) < 3/2$ and $\Delta_{\mathcal{T}}(w', v) < 3/2$, and by Lemma 8 we have $\Delta_{\mathcal{T}}(w, w') < 3/2$; hence $\Delta_{\mathcal{T}}(u, v) < 3/2$. □

It remains to consider the dilation on pairs of points that involve q_2 . We only consider the dilation on pairs of points (u, q_2) where $u \notin \{p_2, p'_2\}$: the dilation of (p_2, q_2) and (p'_2, q_2) depends critically on the choice of standard tree and we will defer its analysis to the next lemma.

Lemma 12 For any vertex $u \in S \setminus \{p_2, p'_2\}$ we have $\Delta_{\mathcal{T}}(u, q_2) < 3/2$.

Proof. We distinguish four cases: first $u = q_1$, second u is on the path from a_1 to a_{n+1} , third $u = d_i$ for some $1 \leq i \leq n$, and finally $u = p_1$ (the cases in which u lies to the left of the y -axis are symmetric).

First, if $u = q_1$, then the connection between u and q_2 is a straight line and the dilation is 1.

Second, if u lies on the path from a_1 to a_{n+1} , let $r = (0, -15/8)$ be the intersection of $q_1 q_2$ with the line through a_1 and a_{n+1} . With the sine rule we get:

$$\begin{aligned} |u^\perp r| + |r q_2| &= \frac{\sin \angle r u^\perp q_2 + \sin \angle u^\perp q_2 r}{\sin \angle q_2 r u^\perp} |u^\perp q_2| \\ &\leq \frac{2 \sin(\frac{1}{2}(\angle r u^\perp q_2 + \angle u^\perp q_2 r))}{\sin \angle q_2 r u^\perp} |u^\perp q_2| \\ &= \frac{2 \sin(\frac{1}{2}(\pi - \angle q_2 r u^\perp))}{\sin \angle q_2 r u^\perp} |u^\perp q_2| \\ &= \frac{2/\sqrt{5}}{4/5} |u^\perp q_2| = \frac{\sqrt{5}}{2} |u^\perp q_2|. \end{aligned}$$

With Lemma 6 we now get:

$$\begin{aligned} \frac{d_{\mathcal{T}}(u, q_2)}{|u q_2|} &\leq \frac{d_{\mathcal{T}}(u, a_1) + d_{\mathcal{T}}(a_1, r) + |r q_2|}{|u^\perp q_2|} \\ &\leq \frac{\frac{22}{21}|u^\perp a_1| + (|a_1 r| + \frac{5}{4}) + |r q_2|}{|u^\perp q_2|} \\ &< \frac{\frac{22}{21}|u^\perp a_1| + |a_1 r| + |r q_2|}{|u^\perp q_2|} + \frac{5/4}{|q_1 q_2|} \\ &< \frac{22}{21} \cdot \frac{|u^\perp r| + |r q_2|}{|u^\perp q_2|} + \frac{5/4}{\frac{25}{9}4^n - \frac{11}{18}} \\ &\leq \frac{11}{21}\sqrt{5} + \frac{5}{42} < \frac{3}{2}. \end{aligned}$$

Third, if $u = d_i$, we get:

$$\begin{aligned}
\frac{d_{\mathcal{T}}(u, q_2)}{|uq_2|} &\leq \frac{d_{\mathcal{T}}(d_i, a_1) + d_{\mathcal{T}}(a_1, r) + |rq_2|}{|d_i^* q_2|} \\
&\leq \frac{(2|d_i^* a_{i+1}| + \frac{22}{21}|d_i^* a_1|) + (|a_1 r| + \frac{5}{4}) + |rq_2|}{|d_i^* q_2|} \\
&= \frac{\frac{22}{21}|d_i^* a_1| + |a_1 r| + |rq_2|}{|d_i^* q_2|} + \frac{2|d_i^* a_{i+1}| + \frac{5}{4}}{|d_i^* q_2|} \\
&< \frac{22}{21} \cdot \frac{|d_i^* r| + |rq_2|}{|d_i^* q_2|} + \frac{4^i + \frac{5}{4}}{\frac{25}{9}4^n - \frac{11}{18} + \frac{27}{10}4^i - 3} \\
&= \frac{11}{21}\sqrt{5} + \frac{3}{10} - \frac{\frac{5}{6}4^n - \frac{7}{3} - \frac{19}{100}4^i}{\frac{25}{9}4^n - \frac{65}{18} + \frac{27}{10}4^i} \\
&< \frac{11}{21}\sqrt{5} + \frac{3}{10} < \frac{3}{2}.
\end{aligned}$$

Finally, if $u = p_1$, we have

$$\begin{aligned}
d_{\mathcal{T}}(u, q_2) &\leq |q_2 q_1| + |q_1 a_1| + |a_1 a_{n+1}| + \sigma + |a_{n+1} p_1| \\
&= \frac{25}{9}4^n - \frac{11}{18} + \frac{5}{2} + 5 \cdot (4^n - 1) \\
&\quad + \frac{1}{10} + \frac{5}{9}4^n - \frac{179}{360} \\
&= \frac{25}{3}4^n - \frac{421}{120} \\
&< 8.34 \cdot 4^n - 3.50.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|uq_2|^2 &= \left(\frac{13}{3}4^n - \frac{1079}{600}\right)^2 + \left(\frac{16}{3}4^n - \frac{241}{75}\right)^2 \\
&= \frac{425}{9}16^n - \frac{1795}{36}4^n + \frac{195257}{14400} \\
&> 6.87^2 \cdot 16^n - 50.5632 \cdot 4^n + 3.68^2 \\
&= (6.87 \cdot 4^n - 3.68)^2 \\
&> \left(\frac{2}{3}(8.34 \cdot 4^n - 3.50)\right)^2,
\end{aligned}$$

and the claim follows. \square

We have now completed our analysis of standard trees. It remains to show that if solution to the PARTITION instance exists, then we can choose a standard tree with dilation $3/2$. Given A, A' with $A \cup A' = \{1, \dots, n\}$, $A \cap A' = \emptyset$, and $\sigma_A = \sigma_{A'} = 1/20$, we construct a standard tree \mathcal{T} as follows: If $i \in A$, then \mathcal{T} contains $c_i d_i$ and $c'_i a'_{i+1}$, otherwise (that is, if $i \in A'$) \mathcal{T} contains $c'_i d'_i$ and $c_i a_{i+1}$.

Lemma 13 *The tree \mathcal{T} constructed above has dilation $3/2$.*

Proof. Lemmas 6 to 12 prove that we have $\Delta_{\mathcal{T}}(u, v) < 3/2$ for all pairs of points $u, v \in S$, except possibly for the pairs (d'_i, d_i) (with $1 \leq i \leq n$), (p_2, q_2) , and (p'_2, q_2) .

By construction, for any $1 \leq i \leq n$ either d_i is on the path from a_{n+1} to a_1 , or d'_i is on the path from a'_{n+1} to a'_1 . Hence $\Delta_{\mathcal{T}}(d'_i, d_i) < 3/2$ by Lemma 9.

It remains to check the dilation of (p_2, q_2) and (p'_2, q_2) . We have:

$$\begin{aligned}
d_{\mathcal{T}}(p_2, q_2) &= |p_2 a_{n+1}| + d_{\mathcal{T}}(a_{n+1}, a_1) + |a_1 q_1| + |q_1 q_2| \\
&= |p_2 a_{n+1}| + |a_{n+1} a_1| + \sigma_A + |a_1 q_1| + |q_1 q_2| \\
&= \frac{5}{9}4^{n+1} - \frac{179}{90} + 5(4^n - 1) + \frac{1}{20} \\
&\quad + \frac{5}{2} + \frac{25}{9}4^n - \frac{11}{18} \\
&= \frac{5}{2}4^{n+1} - \frac{101}{20}.
\end{aligned}$$

Since $|p_2 q_2| = \frac{5}{3}4^{n+1} - \frac{101}{30}$, it follows that $\Delta_{\mathcal{T}}(p_2, q_2) = 3/2$. By a symmetric calculation we can show that $\Delta_{\mathcal{T}}(p'_2, q_2) = 3/2$. \square

3.4 Reduction with integer coordinates

To complete our NP-hardness proof, we need to show that the reduction can be done on a Turing-machine or an equivalent model. The construction in Section 3.1 does not achieve that yet, because the points d_i are defined as the solution of a quadratic equation.

Instead of the points d_i originally defined, we will therefore compute *approximations* \tilde{d}_i with $|d_i - \tilde{d}_i| < \varepsilon$, for an ε to be determined later. We denote by \tilde{S} the set of points obtained that way, that is, the set of points $a_i, b_i, c_i, \tilde{d}_i, p_i$ and their mirror images as well as the two points q_i .

In the following lemma we bound by how much the dilation of the corresponding points in S and \tilde{S} can differ.

Lemma 14 *We have $|\Delta(S) - \Delta(\tilde{S})| < 4^{n+6}n\varepsilon$.*

Proof. Let u, v be a pair of points in S , let \tilde{u}, \tilde{v} be the corresponding points in \tilde{S} , and let \mathcal{T} be any spanning tree on S . By slight abuse of notation, we will allow \mathcal{T} to also denote the corresponding tree on \tilde{S} . We set $X := d_{\mathcal{T}}(u, v)$, $\tilde{X} := d_{\mathcal{T}}(\tilde{u}, \tilde{v})$, $Y := |uv|$, and $\tilde{Y} := |\tilde{u}\tilde{v}|$. Since $|u\tilde{u}| < \varepsilon$ and $|v\tilde{v}| < \varepsilon$, we have $|Y - \tilde{Y}| < 4\varepsilon$. The path from \tilde{u} to \tilde{v} in \mathcal{T} passes at most $2n$ approximated points, and so $|X - \tilde{X}| < 4n\varepsilon$.

The edges of \mathcal{T} have length less than 4^{n+3} , and so $X < (|S| - 1)4^{n+3} \leq 15n \cdot 4^{n+3}$. We have $Y \geq 1$, and $\tilde{Y} \geq 1$, and thus get

$$\begin{aligned}
\frac{\tilde{X}}{\tilde{Y}} - \frac{X}{Y} &= \frac{\tilde{X}Y - X\tilde{Y}}{Y\tilde{Y}} \\
&< \frac{Y(X + 4n\varepsilon) - X(Y - 4\varepsilon)}{Y\tilde{Y}} \\
&= \frac{4n\varepsilon}{\tilde{Y}} + \frac{4\varepsilon X}{Y\tilde{Y}} \\
&\leq 4n\varepsilon + 4\varepsilon(15n4^{n+3}) \leq 4^{n+6}n\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{X}{Y} - \frac{\tilde{X}}{\tilde{Y}} &= \frac{X\tilde{Y} - \tilde{X}Y}{Y\tilde{Y}} \\
&< \frac{X(Y + 4\varepsilon) - \tilde{X}(Y - 4n\varepsilon)}{Y\tilde{Y}} \\
&= \frac{4\varepsilon X}{Y\tilde{Y}} + \frac{4n\varepsilon}{\tilde{Y}} \leq 4^{n+6}n\varepsilon.
\end{aligned}$$

Taken together this implies $|X/Y - \tilde{X}/\tilde{Y}| < 4^{n+6}n\varepsilon$, or

$$|\Delta_{\mathcal{T}}(u, v) - \Delta_{\mathcal{T}}(\tilde{u}, \tilde{v})| < 4^{n+6}n\varepsilon.$$

Since this is true for any pair u, v and any spanning tree \mathcal{T} , the lemma follows. \square

We will choose $\varepsilon < \xi/(4^{n+7}n)$, and so Lemma 14 implies that $|\Delta(S) - \Delta(\tilde{S})| < \xi/4$. We proved in the previous section that if our PARTITION instance has a solution, then $\Delta(S) \leq 3/2$, and therefore $\Delta(\tilde{S}) < 3/2 + \xi/4$. On the other hand, we showed in Section 3.2 that if the PARTITION instance has no solution, then $\Delta(S) \geq 3/2 + \xi$, and so $\Delta(\tilde{S}) > 3/2 + 3\xi/4$. It follows that by determining whether or not $\Delta(\tilde{S}) \leq 3/2 + \xi/2$, we can still decide the correct answer to the PARTITION instance.

It remains to show that the reduction can be done in polynomial time using integer coordinates of polynomial complexity only. Note that the *input size* is

the total bit complexity of the n integers that form our PARTITION instance. We need to ensure that our construction produces integers with total bit complexity polynomial in this input size.

Let k be an integer with $k > 4n + 22 + \log n + \log \dot{\sigma}$. Clearly k is polynomial in the input size, and we have $2^{-k} < \xi/(4^{n+7}n)$. By the above, it suffices to ensure that $|d_i - \tilde{d}_i| \leq 2^{-k}$, that is, it suffices to compute \tilde{d}_i with k bits after the binary point.

We will multiply all coordinates in our construction by $1800 \cdot 2^k$. We first observe that the coordinates of the points p_i, q_i, a_i, b_i, c_i are now all integers, and by the above it suffices to approximate \tilde{d}_i by an integer as well. The largest coordinate is less than $1800 \cdot 2^k \cdot 2 \cdot 4^{n+1}$, so all numbers can be represented with at most $2n + k + 24$ bits. This implies that the total bit complexity of our construction is polynomial in the input size of the PARTITION instance.

The threshold $3/2 + \xi/2$ can be expressed as a rational number P/Q , with $P = 3 \cdot 4^{n+4}\dot{\sigma} + 1$ and $Q = 2 \cdot 4^{n+4}\dot{\sigma}$. Both numbers have bit complexity polynomial in the input size as well.

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