Matching Problems with Delta-Matroid Constraints

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Abstract

Given an undirected graph $G = (V, E)$ and a directed graph $D = (V, A)$, the master/slave matching problem is to find a matching of maximum cardinality in $G$ such that for each arc $(u, v) \in A$ with $u$ being matched, $v$ is also matched. This problem is known to be NP-hard in general, but polynomially solvable in a special case where the maximum size of a connected component of $D$ is at most two.

This paper investigates the master/slave matching problem in terms of delta-matroids, which is a generalization of matroids. We first observe that the above polynomially solvable constraint can be interpreted as a delta-matroid. We then introduce a new class of matching problem with delta-matroid constraints, which can be solved in polynomial time. In addition, we discuss our problem with additional constraints such as capacity constraints.

Keywords: constrained matching, delta-matroid, polynomial-time algorithm, mixed matrix theory

1 Introduction

For an undirected graph $G = (V, E)$, a subset $M$ of $E$ is called a matching if no two edges in $M$ share a common vertex incident to them. The matching problem is a fundamental topic in combinatorial optimization, and many polynomial-time algorithms have been developed (see e.g., [22, 30]). When we apply the matching problem to practical problems such as scheduling, it is often natural to have some additional constraints. In this literature, there are matching problems with a variety of constraints such as matroids [20], trees [9], precedence constraints [1, 18], and knapsack constraints [2]. These constrained problems are known to be NP-hard in many cases except for special cases of matroid constraints.

The main purpose of this paper is to investigate constraints that preserve polynomial solvability of the matching problem. We provide a new constraint in a connection with a delta-matroid, which is a generalization of a matroid.

1.1 Master/slave matching problem

One of constrained matching problems which contain a polynomially solvable case is the master/slave matching problem arising in a manpower scheduling problem in printing plants [18]. In this scheduling problem, jobs are divided into “master jobs” and “slave jobs.” A master job can be done any time, while a slave job has to be done with its master job. The task is to find an optimal assignment of jobs to workers which satisfies such precedence constraints. This problem can be modeled as the master/slave matching problem.

Let us describe the master/slave matching problem formally. Let $G = (V, E)$ be an undirected graph and $D = (V, A)$ be a directed graph with the same vertex set as $G$. For $(u_s, u_m) \in A$, we say that $u_s$ is a slave of $u_m$ and $u_m$ is a master of $u_s$. A master/slave matching (MS-matching for short) is a matching $M$ in $G$ which satisfies the following property:

\[(u_s, u_m) \in A, u_s \in \partial M \Rightarrow u_m \in \partial M, \tag{1}\]

where $\partial M$ denotes the set of vertices incident to edges in $M$. Note that an MS-matching may contain a master but not its slave. Figures 1 and 2 show examples of an MS-matching, where the former depicts an MS-matching in a non-bipartite graph and the latter in a bipartite graph. The MS-matching problem (MSMP) is to find an MS-matching of maximum cardinality. Given a weight function $c : E \rightarrow \mathbb{R}_+$, the weighted MSMP is to find an MS-matching $M$ such that $c(M) := \sum_{e \in M} c(e)$ is maximum. The (weighted) MSMP was first introduced by Hefner and Kleinschmidt [18]. They proved that it is NP-hard in general, while it can be solved in $O(|V|^3)$ time under the assumption that $k(D) \leq 2$, where $k(D)$ denotes the maximum size of a connected component of $D$. Such MSMP is called the 2-MSMP. In

![Figure 1: An example of an MS-matching in a non-bipartite graph, where bold lines show edges in a matching.](image-url)

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where Hefner dealt with a special case of the 2-MSMP (DM) for symmetric exchange axiom nonempty family delta-matroid ordered matroid parity problem. The bipartite 2-MSMP with its min-max theorem was presented by a delta-matroid over the vertex set \( V \). This problem is NP-hard in contrast to the 2-MSMP. A delta-matroid was introduced by Bouchet [5], and a nonnegative diagonal matrix \( P \), and the 2-MSMP as described in Section 3. In Section 4, we show that this problem can be solved in polynomial time if \( L \) is generic, that is, each entry in \( L \) is an independent parameter. This can be done by reducing it to the maximum weight matching problem. In addition, we consider a special case of constraint matching problem with a delta-matroid \((V, F(L) \triangle S)\) where \( F(L) \triangle S = \{F \cup S \mid F \in F(L)\} \) for a subset \( S \subset V \). This problem properly includes the standard matching problem (when \( L \) is the identity matrix and \( S = \emptyset \)), the case where \( F \) is a linear delta-matroid (written \( P = O \)), and the 2-MSMP as described in Section 3. In particular, our class includes a polynomially solvable variant of the MSMP recently introduced by Amanuma and Shigeno [1]. Thus our result enlarges a polynomially solvable class of constrained matching problems. We also note that our class does not include the MSMP itself, because the way of extension from the 2-MSMP is different.

1.2 Delta-matroids

A delta-matroid is a pair \((V, F)\) of a finite set \( V \) and a nonempty family \( F \) of subsets of \( V \) that satisfies the symmetric exchange axiom:

\[(DM) \quad \text{For } F, F' \in F \text{ and } u \in F \triangle F', \text{ there exists } v \in F \triangle F' \text{ such that } F \triangle \{u, v\} \in F,\]

where \( I \triangle J \) denotes the symmetric difference, i.e., \( I \triangle J = (I \cup J) \setminus (I \cap J) \). The set \( V \) is called the ground set, and \( F \) in \( F \) is a feasible set. A delta-matroid \((V, F)\) is said to be even if \( |F \triangle F'| \) is even for all \( F, F' \in F \).

A delta-matroid was introduced by Bouchet [5], and essentially equivalent combinatorial structures were proposed independently by [10; 12]. A delta-matroid is a generalization of matroids, since the families of independent sets and bases of matroids both form delta-matroids. In a similar way to matroids, a greedy algorithm is applicable to maximizing linear functions over a delta-matroid [5].

A simple example of even delta-matroids is the family of the subsets of even size. Another example is the family of vertex sets \( X \) with \( X = \partial M \) for some matching \( M \) in a graph \( G \), called the matching delta-matroid [6]. For a skew-symmetric matrix, the family of column indices that correspond to a nonsingular principal submatrix is known to form an even delta-matroid [4]. A delta-matroid is linear if it can be represented by some skew-symmetric matrix.

1.3 Our contribution

In this paper, we generalize the 2-MSMP in terms of delta-matroids. We first observe that the master/slave constraints in the 2-MSMP can be represented by a delta-matroid over the vertex set \( V \). Indeed, one can easily check that the MS-constraint satisfies the axiom (DM). This observation leads us to new constrained matching problem, called the delta-matroid matching problem, in which, given an undirected graph \( G = (V, E) \) and a delta-matroid \((V, F)\), we aim at finding a maximum cardinality matching \( M \) with \( \partial M \in F \). By the definition, this problem includes the matroid matching problem (see Section 1.4), because a matroid is a delta-matroid. This means that the delta-matroid matching problem can not be solved in polynomial time in general.

Our main purpose of this paper is to investigate a polynomially solvable class of the delta-matroid matching problem. Let \( L \) be a skew-symmetric matrix with nonnegative diagonals, that is, \( L = K + P \) with a skew-symmetric matrix \( K \) and a nonnegative diagonal matrix \( P \), and \( V \) be the row/column set of \( L \). We denote by \( F(L) \) the family of column indices corresponding to nonsingular principal submatrices of \( L \). Then, it is shown in Section 3 that \( F(L) \) forms a projection of some even delta-matroid, and hence a delta-matroid. Consider the delta-matroid matching problem with a delta-matroid \((V, F(L) \triangle S)\), where \( F(L) \triangle S = \{F \cup S \mid F \in F(L)\} \) for a subset \( S \subset V \). This problem properly includes the standard matching problem (when \( L \) is the identity matrix and \( S = \emptyset \)), the case where \( F \) is a linear delta-matroid (written \( P = O \)), and the 2-MSMP as described in Section 3.

1.4 Related works

Matching problem with matroid constraints has been investigated in combinatorial optimization as a common generalization of two well-known polynomially solvable problems: the matroid intersection problem and the matching problem. The matroid matching problem is that, given an undirected graph \( G = (V, E) \) and a matroid \( M \) on \( V \), we aim at finding a maximum cardinality matching \( M \) such that \( \partial M \) is independent for \( M \). This problem is known to be equivalent to the matroid parity problem [23]. Although it is shown to be intractable in the oracle model [21; 27] and NP-hard for matroids with compact representations [25], Lovász provided a min-max formula [25] for the linear matroid matching problem, and a lot of polynomial-time algorithms for the linear matroid matching have been developed [11; 14; 24; 26; 29].

The delta-matroid matching problem introduced in this paper is a natural generalization of the matroid matching problem to delta-matroids. It should be noted that the feasibility problem for delta-matroid matching, i.e., the problem of finding a matching...
$M$ such that $\partial M$ is feasible, already generalizes the matroid matching problem. Indeed, the feasibility problem is equivalent to the \textit{delta-covering problem}, posed by Bouchet [7] as a generalization of the matroid parity problem. For this problem, Geelen et al. [16] provided a polynomial-time algorithm and a min-max theorem if a given delta-matroid is linear. Thus, for linear delta-matroids, we can find a feasible delta-matroid matching in polynomial time. However, the complexity of finding a maximum delta-matroid matching for linear delta-matroids is still unknown to our knowledge. Note that a polynomial-time algorithm in Section 4 is applicable to some class of linear delta-matroids, where a matrix is assumed to be generic.

When a graph is bipartite and a matroid is defined on each class of the vertex bipartition, the matroid matching problem, also called the \textit{independent matching problem} in this case, can be solved in polynomial time for general matroids [20], because this is equivalent to the matroid intersection. However, the delta-matroid matching with a bipartite graph is as hard as that with a general graph, because a general graph version can be viewed as the delta-matroid intersection, which easily follows from the fact that the family of end vertices of matchings in a graph forms a delta-matroid. In Section 5, we discuss a simpler case where a given delta-matroid is on the one vertex side, and show that such case can be solved in polynomial time.

1.5 Organization

The organization of this paper is as follows. In Section 2, we explain delta-matroid theory and mixed matrix theory. We introduce new constraints by using delta-matroids in Section 3. Section 4 presents a polynomial-time algorithm for a generalized 2-MSMP, and Section 5 deals with the bipartite case.

2 Matrices and delta-matroids

2.1 Graphs and matrices

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For a subset $F$ of $E$, we denote the set of vertices incident to edges in $F$ by $\partial F$. For a subset $X$ of $V$, the \textit{induced subgraph} on $X$ is a graph $G[X] = (X, E')$, where $E' \subseteq E$ is the set of edges whose both end vertices are included in $X$.

Throughout this paper, we consider a matrix over any field, e.g., the real field $\mathbb{R}$. For a matrix $K = (K_{ij})$ with row set $R$ and column set $C$, $K[I, J]$ denotes the submatrix with row set $I \subseteq R$ and column set $J \subseteq C$. For a square matrix $K$, we denote a principal submatrix with row/column set $I$ by $K[I]$. A matrix $K$ is called \textit{skew-symmetric} if $K_{ij} = -K_{ji}$ for all $(i, j)$ and all diagonal entries of $K$ are zero. A \textit{generic matrix} is a matrix in which each nonzero entry is an independent parameter. More precisely, a matrix is \textit{generic} if the set of nonzero entries is algebraically independent over some subfield such as $\mathbb{Q}$. A matrix $K$ is \textit{generic skew-symmetric} if $K$ is skew-symmetric and $\{K_{ij} \mid K_{ij} \neq 0, i < j\}$ is algebraically independent. Let $K$ be a generic skew-matrix with row/column set $V$. The \textit{support graph} of $K$ is the undirected graph $H = (V, E_H)$ with $E_H = \{(i, j) \mid K_{ij} \neq 0, i < j\}$. It is well known that the rank of a generic skew-symmetric matrix $K$ is equal to the maximum size of a matching in the support graph $H$ (see e.g., [28, Proposition 7.3.8]).

By applying this fact to each principal submatrix of $K$, we obtain the following lemma.

**Lemma 2.1.** Let $K$ be a generic skew-symmetric matrix with row/column set $V$, and $H = (V, E_H)$ be its support graph. For a subset $X \subseteq V$, $K[X]$ is nonsingular if and only if $H[X]$ has a perfect matching.

2.2 Delta-matroids and greedy algorithms

Recall that a \textit{delta-matroid} is a pair $M = (V, F)$ of a finite set $V$ and a nonempty family $F$ of subsets of $V$ that satisfies the symmetric exchange axiom (DM).

Let $K$ be a skew-symmetric matrix with row/column set $V$. We denote the family of nonsingular principal submatrices by

$$F(K) = \{X \subseteq V \mid \text{rank} K[X] = |X|\}.$$ 

Then $(V, F(K))$ forms an even delta-matroid, where the empty set is feasible [4].

The \textit{twisting} of a delta-matroid $M = (V, F)$ by $X \subseteq V$ is a delta-matroid defined by $M_{\triangle X} = (V, F_{\triangle X})$, where

$$F_{\triangle X} = \{F \setminus X \mid F \in F\}.$$ 

A delta-matroid $M'$ is \textit{equivalent} to $M$ if $M'$ is a twisting of $M$ by some $X \subseteq V$. A delta-matroid $M$ is said to be \textit{linear} if $M$ is equivalent to $M(K)$ for some skew-symmetric matrix $K$. Since $M(K)$ is an even delta-matroid, a linear delta-matroid is an even delta-matroid. For $X \subseteq V$, the \textit{projection} of $M$ on $X$ is defined by $M|X = (V \setminus X, F|X)$, where

$$F|X = \{F \setminus X \mid F \in F\}.$$ 

A projection is also a delta-matroid. Note that projection does not necessarily preserve evenness.

Given a delta-matroid $M = (V, F)$ and a weight function $c : V \to \mathbb{R}$, consider the problem to find $F \in F$ which maximizes $c(F)$. For this problem, Bouchet [5] designed a greedy-type algorithm, and Shioura and Tanaka [31] extended the algorithm to one for \textit{jump systems} [8], which is a generalization of delta-matroids to integral points. Their result implies that the greedy algorithm finds an optimal solution in polynomial time under the assumption that a membership oracle for $M$ is available, that is, we have an oracle to check whether or not a given set $F$ is in $F$.

2.3 Mixed skew-symmetric matrices and delta-covering

Mixed matrix theory, introduced by Murota [28], is one of the most significant application of delta-matroid theory. In this section, we explain a concept of mixed skew-symmetric matrices, which will be used to analyze the bipartite 2-MSMP in Section 5.

Let $F$ be a field and $K$ be a subfield of $F$. A \textit{typical example} is $K = \mathbb{Q}$ and $F = \mathbb{R}$ or $C$. A skew-symmetric matrix $A$ is called a \textit{mixed skew-symmetric matrix} if $A$ is given by $A = Q + T$, where

\begin{align*}
\text{(MP-Q)} & \quad Q \text{ is a skew-symmetric matrix over } K, \quad \text{and} \\
\text{(MP-T)} & \quad T \text{ is a skew-symmetric matrix over } F \text{ such that the set of nonzero entries is algebraically independent over } K.
\end{align*}

The rank of a mixed skew-symmetric matrix is expressed by using delta-matroids.
For a mixed skew-symmetric matrix $A = Q + T$ with row/column set $V$, it holds that
\[
\text{rank } A = \max \{ \text{rank } Q[I] + \text{rank } T[V \setminus I] \mid I \subseteq V \} = \max \{ |F_Q \triangle F_T| \mid F_Q \in \mathcal{F}(Q), F_T \in \mathcal{F}(T) \},
\]
where $M(Q) = (V, \mathcal{F}(Q))$ and $M(T) = (V, \mathcal{F}(T))$ are the linear delta-matroids defined by $Q$ and $T$, respectively.

For a pair of delta-matroids $(V, \mathcal{F}_1)$ and $(V, \mathcal{F}_2)$, the delta-covering problem is to find $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ maximizing $|F_1 \triangle F_2|$. This problem is a generalization of the matroid parity problem, and contains the delta-matroid intersection problem and the delta-matroid partition problem as special cases (see [28]). Since Geelen et al. [16] gave a polynomial-time algorithm for the delta-covering problem for linear delta-matroids, it follows from Theorem 2.2 that the rank of a mixed skew-symmetric matrix $A = Q + T$ can be computed in polynomial time.

If $A$ is nonsingular, we obtain a delta-covering $F_Q \in \mathcal{F}(Q)$ and $F_T \in \mathcal{F}(T)$ such that $F_Q \cap F_T = \emptyset$ and $|V| = |F_Q| + |F_T|$ by Theorem 2.2. This is rewritten as follows.

**Corollary 2.3.** Let $A = Q + T$ be a nonsingular mixed skew-symmetric matrix with row/column set $V$. Then we can find $I \subseteq V$ in polynomial time such that both $Q[I]$ and $T[V \setminus I]$ are nonsingular.

In the case that $A$ is not nonsingular, we can also compute in polynomial time the maximum size of a nonsingular principal submatrix of $A$ containing a given column set as follows.

**Lemma 2.4.** Let $A = Q + T$ be a mixed skew-symmetric matrix with row/column set $V$. For a weight function $c : V \to \mathbb{R}$ and $W \subseteq V$, we can find $X$ which maximizes $c(X)$ subject to $W \subseteq X$ and $\text{rank } A[X] = |X|$ in polynomial time, if exists.

**Proof.** Since $A$ is skew-symmetric, $(V, \mathcal{F}(A))$ is a delta-matroid. We define a weight function $\tilde{c} : V \to \mathbb{R}$ by
\[
\tilde{c}(v) = \begin{cases} N & (v \in W) \\ c(v) & (v \in V \setminus W) \end{cases},
\]
where $N$ is an integer larger than $c_{\text{max}}|V|$ with $c_{\text{max}} = \max_{v \in V} c(v)$. Let $X$ be a maximum weight feasible solution for $(V, \mathcal{F}(A))$ and $\tilde{c}$, which can be obtained by the greedy algorithm. Then $X \supseteq W$ holds if and only if $X$ maximizes $\tilde{c}(X)$ subject to $W \subseteq X$ and $\text{rank } A[X] = |X|$. If $X \nsubseteq W$, there does not exist $Y$ satisfying $W \subseteq Y$ and $\text{rank } A[Y] = |Y|$.

In the greedy algorithm, it is necessary to determine whether a given $X \subseteq V$ is in $\mathcal{F}(A)$ or not. This is equivalent to computing the rank of $A[X]$, which can be done in polynomial time by solving the linear delta-covering problem. Thus a membership oracle for $(V, \mathcal{F}(A))$ is available, which implies that we can find an optimal $X$ in polynomial time.

### 3 Generalization of 2-MS-constraints with delta-matroids

In this section, we introduce a new class of delta-matroids, which generalizes the MS-constraints with $k(D) \leq 2$. Let $L$ be a skew-symmetric matrix with nonnegative diagonals. The row/column set of $L$ is denoted by $V$. The matrix $L = (L_{ij})$ is called generic if the set $\{L_{ij} \mid L_{ij} \neq 0, i \leq j \}$ is algebraically independent.

Define $\mathcal{F}(L)$ as $\{X \subseteq V \mid \text{rank } L[X] = |X|\}$. For a subset $S \subseteq V$, let us denote
\[
\mathcal{F} := \{F \cup X \mid F \in \mathcal{F}(L), X \subseteq W\},
\]
which is represented by a generalized 2-MS-constraint as follows. Let $L$ be a generic skew-symmetric matrix.
whose support graph coincides with $H$, $P = (P_{ij})$ be a generic diagonal matrix given by

$$P_{ji} = \begin{cases} p_i & (i \in W), \\ 0 & (i \in V \setminus W) \end{cases}$$

and $L = K + P$. Then $F = F(L)$ holds.

In the case of $W = \emptyset$, this constraint arises in the maximum cycle subgraph problem in a 2-edge-colored multigraph [3]. A 2-edge-colored multigraph is a graph which has red edges and blue edges. A cycle subgraph is a union of disjoint cycles whose successive edges differ in color. It is known that a maximum cycle subgraph can be found in polynomial time by the maximum matching problem. Let $G_L$ and $G_V$ be the subgraphs consisting of all the red edges and all the blue edges, respectively. We know that the maximum cycle subgraph problem is equivalent to finding a maximum cardinality matching $\bar{M}$ in $G_L$ subject to $\partial M$ is feasible for the matching delta-matroid of $G_V$.

We here provide a simple example. In the graph depicted in Figure 1, $H = (V, E_H)$ is given by $E_H = \{(u_m, u_n), (v_m, v_n)\}$ and $W = \{w, u_m, v_m\}$. In this case, $F$ is represented by a generic skew-symmetric matrix $L = K + P$ with nonnegative diagonals, where

$$K = \begin{pmatrix} w & u_m & v_m & u_s & v_s \\ u_m & 0 & 0 & 0 & 0 \\ v_m & 0 & 0 & 0 & k_1 \\ u_s & 0 & -k_1 & 0 & 0 \\ v_s & 0 & 0 & -k_2 & 0 \end{pmatrix}$$

and

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 0 \end{pmatrix}.$$

**Example 3.4. (size constraint)** We discuss other constraints by using a twisting operation. The following two examples are motivated by a scheduling problem in which the number of assigned workers is limited.

Let us assume that there are $n$ workers $v_1, v_2, \ldots, v_n$ and we have to choose at most one worker among them. This constraint is expressed by $F(L_1) \triangle S_1$, where

$$L_1 = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ p & k_2 & \cdots & k_n \\ \vdots & \vdots & \cdots & \vdots \\ v_n & -k_n & \cdots & 0 \end{pmatrix}$$

and $S_1 = \{v_1\}$.

Then we have

$$F(L_1) \triangle S_1 = \emptyset, \{v_1\}, \{v_2\}, \ldots, \{v_n\}.$$

We give another example. For two integers $m$ and $n$ $(m \leq n)$, let $K_{m,n}$ be a complete bipartite graph with vertex sets $U$ and $V$. We denote the matching delta-matroid of $K_{m,n}$ by $(U \cup V, F_2)$. For $S_2 = U \setminus V$, $F_2 \triangle S_2$ is the family of all the subsets of size $m$. This means that we have to choose exactly $m$ workers among $U \cup V$.

**Table 1:** The distinction among three kinds of constraints.

<table>
<thead>
<tr>
<th>Ex. 1</th>
<th>constraints</th>
<th>feasible set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = (V, A)$</td>
<td>$\emptyset, {a}, {a, b}$</td>
<td>$\emptyset, {a}, {a, b}$</td>
</tr>
<tr>
<td>$L$ is generic</td>
<td>$a \begin{pmatrix} p &amp; k_1 &amp; 0 \ -k_1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\emptyset, {a}, {a, b}$</td>
</tr>
<tr>
<td>$L$ is over $\mathbb{Q}$</td>
<td>$a \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\emptyset, {a}, {a, b}$</td>
</tr>
</tbody>
</table>

It should be noted that the definition (3) does not extend MS-constraints with $k(D) \geq 3$. Examples in Table 1 show distinction among MS-constraints, generalized 2-MS-constraints with generic $L$, and generalized 2-MS-constraints with $L$ over $\mathbb{Q}$, where $S = \emptyset$.

In this section, we prove that $(V, F)$ given by (2) forms a projection of a certain even delta-matroid, and thus a delta-matroid. We first show the following lemma.

**Lemma 3.5.** Let $L = K + P$ be a skew-symmetric matrix with nonnegative diagonals, and $V$ be the row/column set of $L$. Then $L$ is nonsingular if and only if there exists a nonsingular principal submatrix of $K$ containing $K[V \setminus W]$, where $W$ is a row/column set of $P$ corresponding to positive diagonals.

**Proof.** Since $P$ is a diagonal matrix, we have

$$\det L = \sum_{X \subseteq V} \det K[V \setminus X] \cdot \det P[X] = \sum_{X \subseteq W} \det K[V \setminus X] \cdot \det P[X]$$

by the definition of $W$. Figure 3 shows submatrices $K[V \setminus X]$ and $P[X]$. Since $K[V \setminus X]$ is skew-symmetric, $\det K[V \setminus X] \geq 0$ holds. Moreover, we have $\det P[X] > 0$ for $X \subseteq W$. Hence each term of (5) is nonnegative. Thus, $L$ is nonsingular if and only if there exists $X \subseteq W$ such that $K[V \setminus X]$ is nonsingular.

Let $W$ be the row/column set of $P$ corresponding to positive diagonals. The copy of $W$ is denoted by $W_c$. For $X \subseteq W$, $X_c$ denotes a copy of $X$ included
Lemma 3.6 leads to the following.

\[ \text{Lemma 3.6.} \text{ The following (i) and (ii) hold.} \]

(i) For any \( X \subseteq W_c \cup V \) such that \( L[X] \) is nonsingular, \( L[X \cap V] \) is nonsingular.

(ii) For any \( Y \subseteq V \) such that \( L[Y] \) is nonsingular, there exists a nonsingular matrix \( L[Y] \) satisfying \( Y \subseteq W_c \cup V \) and \( \hat{Y} \cap V = Y \).

Proof. We first prove (i). Let us denote \( X \cap W_c \) by \( Z \) and its copy by \( Z \subseteq W \). By the definition of \( L \), it holds that

\[
\det L[X] = \det L[Z, Z_c] \cdot \det L[Z, Z_c] = \det L[X \setminus (Z \cup Z_c)].
\]

Since \( P[W] \) is a diagonal matrix with positive entries, both \( \det L[Z, Z_c] \) and \( \det L[Z, Z_c] \) are nonzero. By \( \det L[X] \neq 0 \), it holds that \( \det L[X \setminus (Z \cup Z_c)] = K[X \setminus (Z \cup Z_c)] \neq 0 \). Since \( K[X \setminus (Z \cup Z_c)] \) contains \( K[X \setminus \hat{W}, \hat{W}] \), \( L[X \setminus \hat{V}] \) is nonsingular by Lemma 3.5.

We next prove (ii). Since \( L[Y] \) is nonsingular, there exists a nonsingular principal submatrix \( K[Y] \) with \( \hat{Y} \cap \hat{W} = Y \cap W \) and \( \hat{Y} \cap \hat{V} = Y \) by Lemma 3.5. We denote \( Y \setminus \hat{Y} \) by \( Z \). Setting \( \hat{Y} = Z_c \cup Y \), we obtain a nonsingular matrix \( L[Y] \) satisfying \( Y \subseteq W_c \cup V \) and \( \hat{Y} \cap V = Y \).

For \( \hat{S} = S \cup W_c \), we define

\[
\hat{F} = (L) \Delta \hat{S} = \{(X \Delta \hat{S}) \mid \text{rank } \hat{L}[X] = |X|, X \subseteq W_c \cup V\}.
\]

The following corollary follows from Theorem 3.7, because a projection of a delta-matroid is also a delta-matroid. Note that the resulting delta-matroid is not necessarily linear.

**Corollary 3.8.** Suppose \( \mathcal{F} \) is given by (2). The pair \((V, \hat{F})\) is a projection of the linear delta-matroid \((W_c \cup V, \mathcal{F})\) on \( W_c \), where \( \hat{F} \) is defined by (8).

Proof. The projection of \((W_c \cup V, \mathcal{F})\) on \( W_c \) is given by \((V, \hat{F}[W_c])\), where

\[
\hat{F}[W_c] = \{F \setminus W_c \mid F \in \hat{F}\} = \{(X \Delta \hat{S}) \setminus W_c \mid \text{rank } \hat{L}[X] = |X|, X \subseteq W_c \cup V\} = \{(X \setminus V) \Delta S \mid \text{rank } \hat{L}[X] = |X|, X \subseteq W_c \cup V\}.
\]

If \( X \subseteq W_c \cup V \) satisfies \( \text{rank } \hat{L}[X] = |X| \), then \( L[X \setminus V] \) is nonsingular by (i) in Lemma 3.6. This implies \( \hat{F}[W_c] \subseteq \mathcal{F} \). Conversely, if \( Y \subseteq V \) satisfies \( \text{rank } \hat{L}[Y] = |Y| \), it follows from (ii) in Lemma 3.6 that there exists \( Y \subseteq W_c \cup V \) such that \( L[Y] \) is nonsingular and \( \hat{Y} \cap V = \emptyset \). Hence, \( \hat{F}[W_c] \subseteq \mathcal{F}[W_c] \).

The simultaneous delta-matroid properly includes even delta-matroids. We can show that the pair \((V, \hat{F})\) given by (2) is a simultaneous delta-matroid.

### 4 Generalized 2-MSMP

In this section, we discuss non-bipartite matching problem with generalized 2-MS-constraints, and show that it can be solved in polynomial time if \( L \) is generic. Note that Examples 3.1, 3.3-3.4 and a polynomially solvable class in Example 3.2 have a generic \( L \).

Thus, matching problems with these constraints can be solved in polynomial time.

Let \( G = (V, E_G) \) be a graph, and \( c : E_G \to \mathbb{R}_+ \) be an edge weight. For a subset \( S \subseteq V \), let \( \mathcal{F} = \mathcal{F}(L) \Delta S \), where \( L = K + P \) is a skew-symmetric matrix with nonnegative diagonals. Note that \((V, \mathcal{F})\) is a delta-matroid by Corollary 3.8. The **generalized 2-MSMP** is the problem to find a matching \( M \subseteq E_G \) in \( G \) which maximizes \( c(M) \) subject to \( \partial M \in \mathcal{F} \).

The main theorem of this section is the following.

**Theorem 4.1.** If a skew-symmetric matrix \( L \) with nonnegative diagonals is generic, the generalized 2-MSMP can be solved in \( O(|V|^3) \) time.

We first reduce the generalized 2-MSMP to finding a maximum weight matching with common end vertices in two undirected graphs. Let \( \hat{L} \) be a generic skew-symmetric matrix defined by (6), and \( \hat{H} = (W_c \cup V, \hat{E}_H) \) be its support graph.

For a delta-matroid \((V, \mathcal{F})\), we have

\[
\mathcal{F} = \{(X \setminus W_c) \Delta S \mid \text{rank } \hat{L}[X] = |X|, X \subseteq W_c \cup V\}
\]
Consider the graph depicted in Figure 4. The graph $\tilde{H}$ of Example 4.2, where bold lines show edges in a matching.

Then $M$ is a matching in $\Gamma$ satisfying $V_G \cup V_H \cup V_S \subseteq \partial M$ and $\gamma(M) = c(M_G)$. Thus, a matching $M$ in $\Gamma$ has a corresponding matching $M_G$ in $G$, and vice versa.

By Lemma 4.3, we reduce the generalized 2-MSMP to maximum weight matching problem in $\Gamma$. Thus, we can solve this problem in $O(|V|^3)$ time [15; 23]. This completes the proof of Theorem 4.1.

5 Generalized 2-MSMP in bipartite graphs

In this section, we discuss a class of the generalized 2-MSMP, called the generalized 2-MSMP in a bipartite graph. Let $L = K + P$ be a skew-symmetric matrix with nonnegative diagonals over $\mathbb{Q}$. Given a bipartite graph $G = (V^+, V^-; E)$ and a delta-matroid $M = (V^-, F(L))$, we find a maximum cardinality matching $M$ in $G$ satisfying $\partial M \cap V^- \in F(L)$. We remark that $L$ is not assumed to be generic, which is different from Section 4. Instead, we assume $S = \emptyset$ in this section.

In Section 5.1, we show the polynomial solvability of this problem with the aid of mixed matrix theory described in Section 2.3. We further impose capacity constraints to the problem in Section 5.2.

5.1 Algorithm via mixed skew-symmetric matrices

We prove the following theorem.

Theorem 5.1. The generalized 2-MSMP in a bipartite graph can be solved in polynomial time.

We show that the generalized 2-MSMP in a bipartite graph can be solved by a greedy algorithm for an even delta-matroid. For a matching $M$ in $G = (V^+, V^-; E)$, we denote $\partial M \cap V^+$ by $\partial^+ M$ and $\partial M \cap V^-$ by $\partial^- M$. A copy of $V^-$ is denoted by $\hat{V}^-$. We also denote a copy of $X \subseteq V^-$ by $\hat{X} \subseteq \hat{V}^-$. Let $T = (T_{ij})$ be a generic matrix given by $T_{ij} \neq 0$ if $(i, j) \in E$ and $T_{ij} = 0$ otherwise. We now construct a matrix $A = \hat{K} + \hat{P}$ with

\[
\hat{K} = \begin{pmatrix}
V^+ & V^- & \hat{V}^- \\
V^- & O & T \\
\hat{V}^- & -T^T & O & I
\end{pmatrix},
\]

\[
\hat{P} = \begin{pmatrix}
V^+ & V^- & \hat{V}^- \\
O & O & O \\
O & O & O \\
O & O & P
\end{pmatrix},
\]

where $I$ denotes an identity matrix. Then the generalized 2-MSMP in a bipartite graph and $A$ are related as follows.

Lemma 5.2. Let $X$ be a subset of $V^+ \cup V^- \cup \hat{V}^-$ which maximizes $|X \cap V^+|$ subject to $X \supseteq V^- \cup \hat{V}^-$ and rank $A[X] = |X|$. Then $G$ has a maximum cardinality matching $M$ with $\partial^- M \in F(L)$ which satisfies $|M| = |X \cap V^+|$.

Proof. Let $X$ satisfy $X \supseteq V^- \cup \hat{V}^-$ and rank $A[X] = |X|$. Then, by Lemma 3.5 applied to $A[X]$, there exists a nonsingular skew-symmetric submatrix $\hat{K}[Y]$ of $\hat{K}[X]$ such that $Y \supseteq (X \cap V^+) \cup V^-$. Since $\hat{K}[Y]$
is skew-symmetric, there exists $X^- \subseteq V^-$ such that $K[X \cap V^+, X^-]$ and $K[X \cap V^-, Y]$ are nonsingular. The nonsingularity of $K[X \cap V^+, X^-]$ implies that $G$ has a matching $M$ with $\partial^- M = X^-$ by Lemma 2.1. Thus we obtain $|M| = |X^-| = |X^+|$. Since $K[X^- \cap Y]$ is nonsingular, $A[X^-]$ is also nonsingular by Lemma 3.5, which implies $\partial^- M = X^- \in \mathcal{F}(L)$.

Next, let $M$ be a matching in $G$ with $\partial^- M \in \mathcal{F}(L)$ and $X = \partial^- M \cup V^- \cup \hat{V}^-$. Then we have $|M| = |X^+|$. We denote $\partial^- M$ by $X^-$. Since $T[X \cap V^+, X^-]$ and $L[X^-]$ are nonsingular, det $A[X]$ has a nonzero expansion term, which implies that $A[X^-]$ is nonsingular by the genericity of $T$.

Thus, $X$ has a corresponding matching $M$ in $G$, and vice versa.

For $L = K + P$, let $W \subseteq \hat{V}^-$ be the row/column set of $L$ which corresponds to positive diagonals of $P$, and $W_c$ be its copy. Since $A$ is in the form of a skew-symmetric matrix with nonnegative diagonals, we define $\hat{A}$ by

$$\hat{A} = \begin{pmatrix} V^+ & V^- & W_c & \hat{V}^- \\ O & T & O & O \\ -T^+ & O & O & I \\ O & O & -I & \hat{L} \end{pmatrix}$$

in a similar way to (6). Then $\hat{A}$ is a mixed skew-symmetric matrix where $\hat{L}$ is the only generic matrix.

We denote the row/column set of $\hat{A}$ by $\hat{V}$. Let $X$ be a subset of $\hat{V}$ maximizing $|X \cap V^+|$ subject to $X \supseteq V^- \cup \hat{V}^-$ and rank $\hat{A}[X] = |X|$. It should be noted that such $X$ exists, because $\hat{A}[V^- \cup \hat{V}^-]$ is nonsingular. By Lemma 3.6, $X \setminus W_c$ maximizes $|X \cap V^+|$ subject to $X \setminus W_c \supseteq V^- \cup \hat{V}^-$ and rank $A[X \setminus W_c] = |X \setminus W_c|$. Hence, Lemma 5.2 implies that $G$ has a maximum cardinality matching $M$ with $\partial^- M \in \mathcal{F}(L)$ which satisfies $|M| = |(X \setminus W_c) \cap V^+|$. By Lemma 2.4, we can find such $X$ by a greedy algorithm for an even delta-matroid $M(\hat{A}) = (\hat{V}, \mathcal{F}(\hat{A}))$ and a weight function $c: \hat{V} \to \mathbb{R}_+$ defined by

$$c(v) = \begin{cases} 1 & (v \in V^+) \\ 0 & (v \in V^- \cup W_c \cup \hat{V}^-) \end{cases}$$

Then we obtain $I$ and $J$ such that $T[I, J]$ is nonsingular by applying Corollary 2.3 to $\hat{A}[X]$. A perfect matching $M$ in $G[I \cup J]$ is an optimal solution of the generalized 2-MSMP in a bipartite graph. Thus, $M$ can be found in polynomial time, which completes the proof of Theorem 5.1.

Remark 5.3. If $L$ is generic, we can reduce the generalized 2-MSMP in a bipartite graph to a non-bipartite matching problem. In this case, the matrix $\hat{A}$ is regarded as a generic skew-symmetric matrix, because $\hat{L}$ can be replaced by a generic diagonal matrix. Let $H(\hat{A})$ be the support graph of $\hat{A}$. By the definition of $\hat{A}$, we have a matching $M_0$ in $H(\hat{A})$ which satisfies $\partial^+ M_0 \supseteq \hat{V}^-$ and $\partial^- M_0 \supseteq V^-$. We can obtain an optimal matching by setting $M_0$ as an initial matching and applying an augmenting path type algorithm. This graph $H(\hat{A})$ is essentially equivalent to a graph $\Gamma$ defined in Section 4.

Remark 5.4. It is shown in [1; 19] that the MSMP with $k(D) = 3$ is NP-hard even if $G$ is bipartite. Since the MS-constraint with $k(D) = 3$ can be represented by a (non-linear) delta-matroid, this fact implies that the delta-matroid matching with a bipartite graph is NP-hard in general.

5.2 Algorithm for capacity-constrained case

We now deal with the generalized 2-MSMP in a bipartite graph with capacity constraints. Suppose that we are additionally given a partition $(V^+, \ldots, V^+_k)$ of $V^+$ and positive integers $r_1, \ldots, r_m$. Our task is to find a maximum cardinality matching $M$ in $G$ subject to $\partial^- M \in \mathcal{F}(L)$ and $|\partial M \cap V^+_i| \leq r_i$ for $1 \leq i \leq m$.

In a similar way to Section 5.1, this problem can be solved by a greedy algorithm for an even delta-matroid as follows. Let us define

$$U = \{u^1_1, \ldots, u^1_{i_1}, u^2_1, \ldots, u^2_{i_2}, \ldots, u^m_1, \ldots, u^m_{i_m}\}$$

and $j \in \hat{V}^+$ be a copy of $i \in V^+$. For $k = 1, \ldots, m$, we denote by $T_k'$ a generic matrix such that $(u^k_i, j)$ entry is nonzero if $j \in V^+_k$ and 0 if $j \not\in V^+_k \cup V^+_i$ for $i = 1, \ldots, r_k$. Let $T'$ be a generic matrix with row set $U$ and column set $\hat{V}^+$ such that $T'(u^k_1, \ldots, u^k_{i_k}), V^+) = T_k'$. We now construct a mixed skew-symmetric matrix

$$\hat{A}' = \begin{pmatrix} U & \hat{V}^+ & V^- & W_c & \hat{V}^- \\ O & T & O & O & O \\ -T^+ & O & I & O & O \\ O & -I & O & O & O \\ O & O & O & \hat{A} \end{pmatrix}.$$ 

Let $\hat{V}'$ denote the row/column set of $\hat{A}'$. Then we can prove the following lemma in a similar way to Section 5.1.

Lemma 5.5. Let $X$ be a subset of $\hat{V}'$ maximizing $|X \cap U|$ subject to $X \supseteq \hat{V}^+ \cup V^+ \cup V^- \cup \hat{V}^-$ and rank $\hat{A}'[X] = |X|$. Then, $G$ has a maximum cardinality matching $M$ with $\partial^- M \in \mathcal{F}(L)$ and $|\partial M \cap V^+_i| \leq r_i$ for $1 \leq i \leq m$ which satisfies $|M| = |X \cap U|$.

Let $X$ be a subset of $\hat{V}'$ defined in Lemma 5.5. Such $X$ is found by applying a greedy algorithm to an even delta-matroid $M(\hat{A}') = (\hat{V}', \mathcal{F}(\hat{A}'))$ and a weight function $c: \hat{V}' \to \mathbb{R}_+$ defined by

$$c(v) = \begin{cases} 1 & (v \in U) \\ 0 & (v \in \hat{V}' \setminus U) \end{cases}$$

We can construct from $X$ an optimal solution $M$ of the generalized 2-MSMP in a bipartite graph with capacity constraints in a similar way to Section 5.1.

Thus we obtain the following theorem.

Theorem 5.6. The generalized 2-MSMP in a bipartite graph with capacity constraints can be solved in polynomial time.

If $L$ is generic, we can reduce the generalized 2-MSMP in a bipartite graph with capacity constraints to a maximum cardinality matching problem in a similar way to Remark 5.3.
6 Concluding remarks

We have generalized the 2-MS-constraints using a skew-symmetric matrix $L$ with nonnegative diagonals, and showed that matching problem with this generalized constraint can be solved in polynomial time when $L$ is generic and when a graph is bipartite. The key observation is that the generalized 2-MS-constraint is a projection of a certain even delta-matroid (Corollary 3.8). Let us remark that the arguments in Sections 4 and 5 are applicable to a projection of generic and linear delta-matroid, respectively, and hence Theorems 4.1 and 5.1 can be extended to such projected delta-matroid cases.

As mentioned in Section 1, our class is a special case of the delta-matroid matching, but unifies previously known classes of polynomially solvable problems such as [1; 3; 18]. An important broader class of the delta-matroid matching is the case where a delta-matroid is linear, which is a generalization of the linear matroid matching and the linear delta-covering. It may be interesting to adapt some efficient algorithms such as [11; 29] to the linear delta-matroid matching.

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References


